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Comments on Dr. Mühlpfort and the determination of Production Prices and the Uniform Rate of Profit for a given Uniform Real Wage Rate*

by
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The short article by Mühlpfort "Karl Marx and the average rate of profit"[1] is for two reasons of exceptional importance for economic science. The *first* is that it constitutes the *first* correct formulation of the Marxian problem of transforming labour values into production prices. Even the title of the article shows that Mühlpfort has correctly understood the problem. For, if one takes into consideration that the *average rate of profit* here means, as compared to the use of the term by Marx, *general or uniform rate of profit*, the title's reference to the average rate of profit instead of, as one would expect, to *production prices*, shows that Mühlpfort correctly considers that the formation of a uniform rate of profit is the same thing as the formation of production prices or, in other words, that in his view, the formation of a uniform rate of profit and the formation of production prices constitute one and the same process. For this reason, he himself in the solution to the problem of transforming labour values into production prices at the same time determines the aforesaid prices and the uniform rate of profit.

The *second* reason is the following: The view generally prevails that, if the theory of inputs-outputs and linear production systems is genetically related to Marx's theory, then it is related to that part pertaining to reproduction, and in particular to reproduction schema, and not to his theory of production prices. This work of his, in which Mühlpfort presents, in order to set out and solve the transformation problem, with the greatest possible clarity and simplicity and in a very modern way, a point input point output model of production, shows, just

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* I would like to thank two anonymous referees for helpful comments. The remaining errors are mine.

1. Muhlpfort's article, which was published in 1895, went unnoticed until 1987, when M. C. Howard and J. E. King presented it in a brief note (see Howard and King (1987)).

as Charasoff's paper on the same subject which was published 15 years later[2], that this is not the way things are and that the first papers written on systems of inputs-outputs and linear production systems have as their starting point the occupation of their respective authors with the Marxian problem of transforming labour values into production prices. Dmitriev's paper on Ricardo is also in this direction: and Dmitriev, as early as 1898, formulates a "flow input-point output" model of production, treating the problem of the formation of production prices in its Ricardian context. Generalising, we can say that the starting point of the first formulations of the theory of linear production systems at the end of the 19th century was not speculation about production and reproduction, but problems relating to the theory of prices. And this is true not only with respect to analyses that had Marx or Ricardo as their starting point, but also those of neoclassical economists, such as those of Walras.

We shall disregard Mühlpfort's views on the classical theory of labour value and prices and its compatibility with the corresponding theory of the Austrian school (neoclassical theory); we shall also disregard his evaluation of the Marxian solution to the problem of transforming labour values into production prices[3] and we shall begin with the presentation of the formulation and solution of the problem which he himself puts forward. We should however like to dwell on a certain point in the treatment of the transformation problem by Marx, to which Mühlpfort also refers, although somewhat inadequately. Namely, Marx's observation that if one sets, as he himself does in the solution to the problem, the cost-prices of commodities not at production prices but labour values, then the solution is not faultless. Marx writes:

"However... a modification appears [in the method of calculating production prices – G.S.] concerning the determination of the cost-price of commodities. We had originally assumed that the cost-price of a commodity equalled the *value* of the commodities consumed in its production. But for the buyer the price of production of a specific commodity is its cost-price, and may thus pass as cost-price into the prices of other commodities. Since the price of production may differ from the value of a commodity, it follows that the cost-price of a commodity containing this price of production of another commodity may also stand above or below that portion of its total value derived from the value of the means of production [and wage commodities – G.S.] consumed by

2. See Charasoff (1910), Stamatis (1988) and Stamatis (1999).

3. For the evaluation of the Marxian solution see Stamatis (1995).

it. It is necessary to remember this modified significance of the cost-price, and to bear in mind that *there is always the possibility of an error if the cost-price of a commodity in any particular sphere is identified with the value of the means of production* [in which we also include wage commodities – G.S.] *consumed by it*. Our present analysis does not necessitate a closer examination of this point"[4].

In reality, by virtue of this observation Marx has already correctly set forth the problem. Thus, the correct formulation (not the solution) of the problem by both Mühlpfort and von Bortkiewicz, as well as by Charasoff, is in actual fact its Marxian formulation. This must be stated, because none of the aforementioned persons acknowledges this debt to Marx, but rather each of them presents the correct formulation of the problem (which solely in this specific respect had difficulties) as exclusively their own achievement. It thus becomes understandable why all those who attempted to solve the problem (Lexis, Fireman, Conrad Schmidt) prior to the publication of the third volume of *Das Kapital* did not think to do what they who solved it after the publication of the third volume of *Das Kapital* did, namely to set cost-prices at production prices and not labour values: because this first became known from the respective analyses of Marx himself in volume III of *Das Kapital*.

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Before mathematically setting the price of production, on the basis of his formulation that the price of production deviates from the value of a commodity, Mühlpfort puts forward a relation between the price of production and the value of a commodity. Thus, if the price of production of one unit of any random commodity W is $\Pi(W)$ and the value of one unit of the same commodity a_w, then the following holds

$$\Pi(W) = a_w x_w,$$

where x_w is a coefficient, which of course is not the same for all the commodities (because otherwise production prices would be proportional to values and the relative production prices would not deviate from the corresponding relative values). Mühlpfort defines the price of production of one unit of commodity W with the equation

$$\Pi(W)=\Pi(C)+\Pi(M),$$

4. Karl Marx (1867), p. 174. Mühlpfort avoids citing this excerpt.

where $\Pi(C)$ is the cost-price, calculated in *production prices*, of one unit of the commodity W and $\Pi(M)$ the profit, also expressed in *production prices*, which is contained in the price of production of one unit of commodity W.

Because the exchange of commodities at production prices entails the formation of a uniform rate of profit, which shares the total profit of the economy among the various commodities in proportion to the capital (constant and variable) which was used in their production, with respect to the profit $\Pi(M)$ which is contained in the price of production of one unit of commodity W, and on the condition that the constant capital is entirely used up during the period of production, the following holds:

$$\Pi(M) = p\Pi(C),$$

where p is the uniform rate of profit. Consequently, the following holds for the price of production of one unit of commodity W

$$\begin{aligned}\Pi(W) &= \Pi(C) + \Pi(M)\\ &= \Pi(C) + p\Pi(C)\\ &= \Pi(C)(1 + p)\end{aligned}$$

and for the cost-price of one unit of commodity W

$$\Pi(C) = \frac{1}{1+p}\Pi(W) = x_0\Pi(W),$$

where

$$x_0 = \frac{1}{1+p}.$$

Lastly, because of $\Pi(W)=a_w x_w$, the following holds for the cost-price of one unit of commodity W

$$\Pi(C) = x_0 x_w a_w$$

and, correspondingly, for the cost-price of one unit of commodity p

$$\Pi(C_p) = x_0 x_p a_p.$$

In this equation for determining the cost-price of one unit of commodity p, i.e. in the equation for determining the price of production of the constant and variable capital expended (of the used-up means of production and, indirectly, of the wage commodities expended) in the production of one unit of commodity p, this capital no longer appears as a commodity necessary for the production of another commodity but rather as a *produced commodity* itself. One may also present the price of production $\Pi(C_p)$ of this capital as the sum of the price of production of the means of production that were used up and of the price of production of the wage commodities, which, indirectly, were expended for its production, in which case we get

$$\Pi(C_p) = \alpha_{p1}\Pi(W_1) + \alpha_{p2}\Pi(W_2) + \ldots + \alpha_{pn}\Pi(W_n),$$

where $\alpha_{p1}, \alpha_{p2}, \ldots, \alpha_{pn}$ are the quantities of commodities $W_1, W_2, \ldots, W_n$, that is, of the means of production that were used up and wage commodities expended for the production of this constant and variable capital C_p (=of the constant and variable capital necessary for the production of one unit of commodity p).

From this last equation and $\Pi(C_p) = x_0\Pi(W_p)$ we get

$$x_0\Pi(W_p) = \alpha_{p1}\Pi(W_1) + \alpha_{p2}\Pi(W_2) + \ldots + \alpha_{pn}\Pi(W_n).$$

This equation holds for the capital expended for the production of one unit of commodity p. For the capitals expended for the production of each of the produced commodities, the following equations hold respectively

$$\begin{aligned} x_0\Pi(W_1) &= \alpha_{11}\Pi(W_1) + \alpha_{12}\Pi(W_2) + \ldots + \alpha_{1n}\Pi(W_n) \\ x_0\Pi(W_2) &= \alpha_{21}\Pi(W_1) + \alpha_{22}\Pi(W_2) + \ldots + \alpha_{2n}\Pi(W_n) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_0\Pi(W_n) &= \alpha_{n1}\Pi(W_1) + \alpha_{n2}\Pi(W_2) + \ldots + \alpha_{nn}\Pi(W_n) \end{aligned}$$

or

$$\begin{aligned} x_0a_1x_1 &= \alpha_{11}a_1x_1 + \alpha_{12}a_2x_2 + \ldots + \alpha_{1n}a_nx_n \\ x_0a_2x_2 &= \alpha_{21}a_1x_1 + \alpha_{22}a_2x_2 + \ldots + \alpha_{2n}a_nx_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_0a_nx_n &= \alpha_{n1}a_1x_1 + \alpha_{n2}a_2x_2 + \ldots + \alpha_{nn}a_nx_n. \end{aligned}$$

This system of equations may be written with the help of matrices and vectors as follows:

$$x_0\beta = A\beta,$$

where β,

$$\beta = \begin{bmatrix} a_1 x_1 \\ a_2 x_2 \\ \vdots \\ a_n x_n \end{bmatrix}$$

is the column vector of the production prices of n commodities and A,

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}$$

the matrix of technical coefficients, each *row* of which contains as elements the inputs in means of production and wage commodities necessary for the production of one unit of the respective commodity, and therefore describes, according to Mühlpfort, «the technique of the respective enterprise». So matrix A describes the «technologies» (Mühlpfort) of all the enterprises (or branches), each of which produces one and only one (different to those produced by the rest) of the n commodities. We would say today that it describes the given linear technique of the production system.

From the last formulation of the system of equations we get

$$(x_0 I - A)\beta = 0.$$

As we shall see later, this system of equations determines the uniform rate of profit and the n-1 *relative* production prices of n commodities, i.e. the n-1 ratios of n *absolute* production prices of the commodities. In order to determine the uniform rate of profit and the n absolute prices of commodities, Mühlpfort introduces a normalisation equation (of the vector) of production prices, setting the unknown total profit of the system equal to the – known – total surplus value of the system and thus the ratio of total profit to total surplus value equal to unit:

$$\Sigma\Pi = \Sigma\alpha,$$

where Π is profit and α the surplus value per unit of commodity[5].

5. Howard and King (1987), p. 266, misconstrue it as the price of production.

Let's see how we can write this last equation. The total surplus value is equal to the value of the surplus product of the overall system and the total profit equal to the price of production of the surplus product of the overall system. So in order to calculate these two magnitudes, we must first calculate the total surplus product of the system. The gross product of the system, which consists of one unit of each of the n commodities[6], is therefore

$$s = [1, 1, ..., 1].$$

Consequently, for the surplus product U we get

$$U = s - sA = s(I - A). \tag{1}$$

For A we write

$$A = \bar{A} + \ell d, \tag{2}$$

where $\bar{A}$, $\bar{A} \geq 0$ the matrix of inputs in used-up means of production per unit of produced commodity, ℓ, $\ell > 0$, the vector of inputs in direct labour per unit of produced commodity, d, $d \geq 0$ the vector of the given real wage rate and therefore, ℓd, $\ell d \geq 0$ the matrix of inputs in real wages per unit of produced commodity.

We assume that

$$(0 <)\lambda_m^{\bar{A}} < 1 \tag{3}$$

where $\lambda_m^{\bar{A}}$ the Perron-Frobenius (maximum) eigenvalue of $\bar{A}$, and consequently

$$(I - \bar{A})^{-1} \geq 0. \tag{4}$$

This assumption means that the given technique $[\bar{A}, \ell]$ is productive, i.e. that it is able –using some or all the production processes at positive activity levels and none at a negative activity level, so that $z \geq 0$, where z is the vector of activity levels– to produce every exogenously given positive or semi-positive net product Y, $Y \geq 0$[7]. Because here production is single production, vector z of

6. Howard and King do not understand that Mühlpfort –as very often happens today, see for example Sraffa– assumes that each sector produces only one unit of a commodity, or, put differently, that he reduces to a unit of measurement of each commodity the total produced quantity of that commodity.
7. As is known, the following holds

$$z = (I - \bar{A})^{-1} Y.$$

It follows from this relation, taking into consideration (4) that for each $Y \geq 0$, $z \geq 0$.

activity levels at the same time also represents the gross product of the system. According to Mühlpfort, z = s.

For Y the following holds

$$Y = s - s\bar{A} = s(I - \bar{A}). \tag{5}$$

The fact that the technique $[\bar{A}, \ell]$ is productive does not mean that the net product Y of the system is necessarily positive or semi-positive ($Y \geq 0$). As is immediately clear from (5), the net product Y may contain –apart from positive or positive and zero– also negative quantities of commodities. But under no circumstances can it be zero ($Y = 0$). Because in order for $Y = 0$, (5) would give the following:

$$s = s\bar{A}$$

and consequently, according to the known theorem,

$$\lambda_m^{\bar{A}} = 1,$$

which would conflict with (3) and mean that the given technique $[\bar{A}, \ell]$ is not productive. For all the more reason, the net product Y cannot contain only zero and negative or only negative quantities of commodities. Because then, (5) would in the former case give the following

$$s \leq s\bar{A}$$

and consequently

$$\lambda_m^{\bar{A}} \geq 1$$

and in the latter case

$$s < s\bar{A}$$

and consequently

$$\lambda_m^{\bar{A}} > 1$$

which would conflict with (3) and mean that the technique $[\bar{A}, \ell]$ is unproductive.

When the net product contains also negative quantities of commodities, then these negative quantities of commodities are those quantities of means of production that the system used but did not produce in the given period. They are, that is, the quantities of commodities that the system lacks, in order to be able to operate and exist in the given period. And because, by assumption, the system does operate and exist in the given period, this supposition means that the system obtains the aforesaid quantities of means of production, which it cannot get from its current production, from its stocks and/or from other systems with which it has exchanges.

We therefore assume that the given system of production $[\bar{A},\ell,z]$, for which $z=s$, if its net product contains also negative quantities of commodities, obtains these quantities of commodities in the manner set out above.

For values a we get

$$a=\bar{A}a+l \Rightarrow$$

$$(I-\bar{A})a=\ell \Rightarrow \tag{6}$$

$$a=(I-\bar{A})^{-1}\ell. \tag{7}$$

As a consequence of $\ell > 0$ and (4), $a > 0$, i.e. the values of all the commodities are positive.

We also assume that the Perron-Frobenius (maximum) eigenvalue λ_m^{ad},

$$\lambda_m^{ad}=da,\ ^{8}$$

of the matrix ad is smaller than unit:

$$(0<)\lambda_m^{ad}=da<1 \tag{8}$$

and consequently

$$(I-ad)^{-1}\geq 0. \tag{9}$$

The magnitude da clearly represents the value of one unit of labour power. Consequently, by virtue of da we assume that the value of one unit of labour power is smaller than unit.

8. Because the matrix ad follows from the product of two vectors, its maximum eigenvalue is equal to the internal product da of these two vectors and all its other eigenvalues are equal to zero.

For $(I-A)$ we get, taking into consideration (2) and (6),

$$(I-A)=(I-\bar{A})-\ell d=(I-\bar{A})-(I-\bar{A})ad=$$
$$=(I-\bar{A})(I-ad).$$

Thus for $(I-A)^{-1}$ we get:

$$(I-A)^{-1}=[(I-\bar{A})(I-ad)]^{-1}=$$
$$=(I-ad)^{-1}(I-\bar{A})^{-1}.$$

As a consequence of (4) and (9)

$$(I-A)^{-1}\geq 0 \tag{10}$$

and

$$(0<)\lambda_m^A<1 \tag{11}$$

where λ_m^A the maximum eigenvalue of A. (10) and (11) mean that the given technique $[\bar{A},\ell]$ is, for the given real wage rate d, surplus productive, that is to say, it is able –using some or all the production processes at positive activity levels and none at a negative activity level, so that $z \geq 0$– to produce each exogenously given positive or semi-positive surplus product U, $U \geq 0$.[9]

The fact that the technique $[\bar{A},\ell]$ for each given real wage rate d, i.e. eventually technique A, is by assumption surplus productive, does not of course entail that the surplus product U of the given system of production [A, z], for which according to Mühlpfort $z = s$, is necessarily positive or semi-positive, $U\geq 0$. As is immediately clear from (1), the net product Y may contain –apart from positive or positive and zero– also negative quantities of commodities. But under no circumstances can it be zero, $U = 0$. Because in order for $U = 0$, (1) would give the following:

$$s = sA$$

and consequently

9. As is known, the following holds

$$z = (I-A)^{-1}U.$$

It follows from this relation, taking into consideration (10), that for each $U \geq 0$, $z \geq 0$.

$$\lambda_m^A = 1$$

which would conflict with (11) and mean that the given technique A is not surplus productive. For all the more reason, the surplus product U cannot contain only zero and negative or only negative quantities of commodities. Because then, (1) would in the former case give the following

$$s \leq sA$$

and consequently

$$\lambda_m^A \geq 1$$

and in the latter case

$$s < sA$$

and consequently

$$\lambda_m^A > 1$$

which would conflict with (11) and mean that the given technique A is not surplus productive.

When the surplus product U contains *also* negative quantities of commodities, then these are those quantities of means of production and wage commodities that the system used but did not produce in the given period. They are, that is, the quantities of commodities that the system lacks, in order to be able to operate and exist in the given period. And because, by assumption, the system does operate and exist in the given period, this supposition means that the system obtains these quantities of commodities, which it cannot get from its current production, from its stocks and/or from other systems of production with which it has exchanges. We therefore assume that the given system of production [A, z], for which $z = s$, if its surplus product contains also negative quantities of commodities, obtains these quantities of commodities which it lacks in the manner set out above.

We saw that the labour values of all the commodities are positive. Consequently, when $U \geq 0$, then $Ua > 0$, i.e. surplus value is positive.

Let us look now at the case in which the surplus product U contains also negative quantities of commodities. We shall show that in this case too, surplus value Ua is always positive.

Proof: For surplus value Ua we get, taking into consideration (1) and (7):

$$
\begin{aligned}
\mathrm{Ua} &= \mathrm{U}(\mathrm{I}-\bar{\mathrm{A}})^{-1}\ell \\
&= \mathrm{s}(\mathrm{I}-\mathrm{A})(\mathrm{I}-\bar{\mathrm{A}})^{-1}\ell \\
&= \mathrm{s}(\mathrm{I}-\bar{\mathrm{A}}-\ell\mathrm{d})(\mathrm{I}-\bar{\mathrm{A}})^{-1}\ell \\
&= \mathrm{s}(\mathrm{I}-\bar{\mathrm{A}})(\mathrm{I}-\bar{\mathrm{A}})^{-1}\ell-\mathrm{s}\ell\mathrm{d}(\mathrm{I}-\bar{\mathrm{A}})^{-1}\ell \\
&= \mathrm{s}\ell-\mathrm{s}\ell\mathrm{d}(\mathrm{I}-\bar{\mathrm{A}})^{-1}\ell \\
&= \mathrm{s}\ell-\mathrm{s}\ell\mathrm{da} \\
&= \mathrm{s}\ell(1-\mathrm{da}).
\end{aligned} \tag{12}
$$

Taking into consideration (8), we get from (12)

$$\mathrm{Ua} > 0. \tag{13}$$

Consequently, surplus value is always positive, regardless of whether the surplus product contains also negative quantities of commodities or not.

From (1) and (13) we get for the surplus value Ua:

$$\mathrm{Ua} = \mathrm{s}(\mathrm{I}-\mathrm{A})\mathrm{a} > 0. \tag{14}$$

For total profits Uβ, the following clearly holds:

$$\mathrm{U}\beta = \mathrm{s}(\mathrm{I}-\mathrm{A})\beta. \tag{15}$$

From (15) we get

$$
\begin{aligned}
\mathrm{U}\beta &= \mathrm{s}\beta-\mathrm{sA}\beta \\
&= (1+\mathrm{p})\mathrm{sA}\beta-\mathrm{sA}\beta \\
&= \mathrm{sA}\beta+\mathrm{psA}\beta-\mathrm{sA}\beta \\
&= \mathrm{psA}\beta.
\end{aligned} \tag{16}
$$

For sA, sA>0 clearly holds[10]. As we shall show further on, the vector β of *absolute* production prices is not *always* positive or semi-positive. Consequently, as follows from (16), the profits Uβ –even if the surplus product U is positive or semi-positive, U ≥ 0– are not *always* positive.

Consequently, as follows from (14) and (15), Mühlpfort's normalisation equation ΣΠ=Σα takes the form

10. sA>0 and not sA≥0, because there is no production process into which no commodity does not directly enter. This is a consequence of the fact that d≥0 and ℓ>0.

$$s(I - A)\beta = s(I - A)a > 0, \tag{17}$$

where s(I–A)a = a positive constant. Thus, by virtue of the normalisation equation $\Sigma\Pi=\Sigma\alpha$, Mühlpfort sets the total profits $\Sigma\Pi$, i.e. the price s(I–A)β of the surplus product s(I–A), equal to a positive constant and specifically, equal to the given positive value s(I–A)a of the surplus product s(I–A). So in Mühlpfort, the surplus product of the system functions as a normalisation commodity. For this reason and this reason only, profits are *always* positive in Mühlpfort.

The fact that the surplus product U may contain also negative quantities of commodities does not mean that Mühlpfort's normalisation cannot be applied. It may be applied even when the surplus product U and consequently the normalisation commodity of Mühlpfort's normalisation contain also negative quantities of commodities. The reason for this lies in the fact that (3) and (8) hold and as a consequence, the surplus value Ua is positive also in the case in which the surplus product U contains also negative components. This of course does not mean that it is expedient for one to choose as a normalisation commodity one that contains *also* negative quantities of commodities. Each normalisation commodity, which contains only negative quantities of commodities is of course impermissible, because it arbitrarily entails the negativeness of at least one price of the commodities which are contained in that normalisation commodity, that is, such a normalisation arbitrarily premises the negativeness of the price of at least one commodity.[11]

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11. If, in the case that A is *indecomposable*, one chooses a random –and therefore without any economic significance– normalisation commodity that contains, in addition to positive or positive and zero quantities, also negative quantities of commodities, then either all the prices are positive or all the prices are negative or the system for determining prices and the rate of profit (which contains the aforesaid normalisation equation) is incompatible.
If, in the case that A is *indecomposable*, one uses as a normalisation commodity the surplus product U of the given system of production and consequently normalises prices by means of

$$\Pi=U\beta(=psA\beta)=\theta,\ \theta=\text{positive constant},$$

then, all the *absolute prices* of commodities which result are positive (that is, $\beta > 0$) – *even* if U contains, in addition to positive or positive and zero quantities, also negative quantities of commodities.

Proof: Let $\bar{\beta}$ be the vector of *relative* prices, for which as is known $\bar{\beta} > 0$ holds. Also, because the technique is surplus productive, $p > 0$ and, because A is indecomposable, $sA>0$.

According to that set out above, the system of equations for determining the general rate of profit and the absolute production prices is provided by the following two relations

$$(x_0 I - A)\beta = 0 \tag{18}$$

$$s(I - A)\beta = s(I - A)a. \tag{19}$$

(18) is a homogeneous system of n equations with n+1 unknowns, x_0 and n *absolute* production prices. As is known, it determines x_0 (and consequently the general rate of profit p) and the n-1 *relative* production prices of n commodities. We shall describe its solution for the following cases:

Case 1: A is indecomposable and consequently all the commodities are reproductive commodities, i.e. commodities that directly or indirectly enter the production of all the commodities *including the commodity "labour power"*.

Case 2: A is decomposable and consequently certain commodities are non-reproductive commodities. However, these non-reproductive commodities do not directly or indirectly enter the production of non-reproductive commodities.

Case 3: A is decomposable and the non-reproductive commodities enter the production of non-reproductive commodities in such a proportion that the Perron-Frobenius eigenvalue of the matrix of inputs of the reproductive subsystem is *smaller* than the Perron-Frobenius eigenvalue of the matrix of inputs in non-reproductive commodities of the non-reproductive subsystem.

Case 4: A is decomposable and the non-reproductive commodities enter the production of non-reproductive commodities in such a proportion that the Perron-Frobenius eigenvalue of the matrix of inputs of the reproductive subsystem is *equal* to the Perron-Frobenius eigenvalue of the matrix of inputs in non-reproductive commodities of the non-reproductive subsystem. And

Case 5: A is decomposable and the non-reproductive commodities enter the production of non-reproductive commodities in such a proportion that the Perron-Frobenius eigenvalue of the matrix of inputs of the reproductive subsystem is *greater* than the Perron-Frobenius eigenvalue of the matrix of inputs in non-reproductive commodities of the non-reproductive subsystem.

Lastly, $\beta = \gamma \bar{\beta}$ clearly holds, where γ is a real number. Thus, from the above normalisation equation we get:

$$psA\gamma\bar{\beta} = \theta\,(>0) \Rightarrow \gamma > 0 \Rightarrow \beta > 0$$

We shall leave to the reader the investigation of the cases in which A is *decomposable*.

These five cases exhaust all the possible cases when, as we have assumed here in order to simplify the issue, the system of production is split into only two subsystems, the reproductive and the non-reproductive subsystems. (In the general case however, the non-reproductive subsystem may split into two or more non-reproductive subsystems, with the result that the overall subsystem can split into three or more subsystems).

Case 1

As can clearly be seen from (18) if one writes it in the form

$$x_0\beta = A\beta,$$

$x_0[=1/(1+p)]$ depicts the n eigenvalues of A and β the n right eigenvectors of A. Consequently, (18) gives n solutions for x_0 (and consequently for p) and for β.

As is known, according to the relevant theorem of Frobenius, in the case that, as here, A is indecomposable, to the maximum eigenvalue $x_0^{(m)}$ of A,

$$(0<)x_0^{(m)}\left(=\frac{1}{1+p^{(m)}}\right)=\lambda_m^A(<1), \tag{20}$$

which because of the indecomposability of A is non-recurring –and only to it– there corresponds a strictly positive vector of *relative* production prices $\beta^{(m)}$, $\beta^{(m)}>0$. Because of (20) the arithmetical value $p^{(m)}$ of p which corresponds to $x_0^{(m)}$ is also positive. This arithmetical value $p^{(m)}$ of p is obviously the smallest positive of all the n arithmetical values of p.

Thus the solution of (18), which gives

$$x_0=x_0^{(m)}=\lambda_m^A$$

and consequently

$$p = p^{(m)} > 0$$

and

$$\beta = \beta^{(m)} > 0$$

is the only one that is economically significant.

In *Cases 2, 3, 4 and 5*, i.e. the cases in which non-reproductive commodities are also produced and consequently A is decomposable, the «canonical» form of A is

$$A=\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \text{ with } A_{11} \geq 0,\ A_{21} \geq 0 \text{ and } A_{22} \geqq 0.$$

The importance of the submatrices of A is clear. Submatrix A_{11} is square and indecomposable. Submatrix A_{22} is also, in the case of $A_{22} \geq 0$, square and indecomposable.

Let us now look at *Cases 2, 3, 4 and 5.*

Case 2

In this case, $A_{22} = 0$ and consequently

$$\lambda_m^A = \lambda_m^{A_{11}}$$

From (18) we get

$$x_0 \begin{pmatrix} \beta_I \\ \beta_{II} \end{pmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix} \begin{pmatrix} \beta_I \\ \beta_{II} \end{pmatrix} \Rightarrow$$

$$x_0 \beta_I = A_{11} \beta_I \tag{18a}$$

$$x_0 \beta_{II} = A_{21} \beta_I \tag{18b}$$

with $$\beta = \begin{pmatrix} \beta_I \\ \beta_{II} \end{pmatrix},$$

where β_I is the vector of relative prices of the reproductive commodities and β_{II} the vector of relative prices of non-reproductive commodities. The economically significant solution of (18a) clearly gives

$$x_0 = x_0^{(m)} = \lambda_m^A = \lambda_m^{A_{11}}$$

and consequently

$$p = p_I^{(m)} = \frac{1-\lambda_m^{A_{11}}}{\lambda_m^{A_{11}}} > 0$$

and

$$\beta_{I}=\beta_{I}^{(m)}>0,$$

where $p_{I}^{(m)}$ is the rate of profit of the reproductive subsystem for $\beta_{I}=\beta_{I}^{(m)}>0$.
For $x_0=x_0^{(m)}=\lambda_m^{A}=\lambda_m^{A_{11}}$ and $\beta_{I}=\beta_{I}^{(m)}>0$, (18b) gives

$$\beta_{II}=\beta_{II}^{(m)}>0.$$

Consequently, the economically significant solution given by (18) in the given *Case 2* is, with respect to the positiveness of p and of β, the same as that given by (18) for *Case 1*: In both these cases, not only the rate of profit p but also the relative production prices are positive magnitudes. However, in *Case 2* given here, the uniform rate of profit of the overall system is determined as the uniform rate of profit of the reproductive subsystem $(p^{(m)}=p_{I}^{(m)})$. This of course is also true in *Case 1*. However, in *Case 1*, there is no non-reproductive subsystem. Therefore, in the given *Case 2*, the uniform rate of profit of the reproductive subsystem is valid as a uniform rate of profit of the overall system.

Case 3

Here, $A_{22}\geq 0$, $\lambda_m^{A_{11}}<\lambda_m^{A_{22}}$ and consequently

$$\lambda_m^{A}=\max(\lambda_m^{A_{11}},\lambda_m^{A_{22}})=\lambda_m^{A_{22}}.$$

From (18) we get:

$$\begin{cases} x_0\beta_{I}=A_{11}\beta_{I} & \text{(18a)} \\ x_0\beta_{II}=A_{21}\beta_{I}+A_{22}\beta_{II} & \text{(18c)} \end{cases}$$

In the given *Case 3*, we have two solutions of (18), i.e. of (18a) and (18c).
First solution: The economically significant solution of (18a) is, as we saw above, the following:

$$x_0=x_0^{(m)}=\lambda_m^{A}=\lambda_m^{A_{11}} \;\Rightarrow\; p=p_{I}^{(m)}>0,$$
$$\beta_{I}=\beta_{I}^{(m)}>0.$$

Given this solution of (18a), we get from (18c)

$$\beta_{II} = \frac{1}{\lambda_m^{A_{11}}} A_{21}\beta_I + \frac{1}{\lambda_m^{A_{11}}} A_{22}\beta_{II} \;\Rightarrow$$

$$(I - \frac{1}{\lambda_m^{A_{11}}} A_{22})\beta_{II} = \frac{1}{\lambda_m^{A_{11}}} A_{21}\beta_I \;\Rightarrow$$

$$\beta_{II} = (I - \frac{1}{\lambda_m^{A_{11}}} A_{22})^{-1} \frac{1}{\lambda_m^{A_{11}}} A_{21}\beta_I. \tag{18cc}$$

From

$$(I - \frac{1}{\lambda_m^{A_{11}}} A_{22})\beta_{II} = \frac{1}{\lambda_m^{A_{11}}} A_{21}\beta_I \;\Rightarrow$$

we get

$$\beta_{II} \ngeq 0,$$

i.e. that one or some or all the prices of the non-reproductive commodities are necessarily negative.

Proof: Because $\frac{1}{\lambda_m^{A_{11}}} A_{21}\beta_I \geq 0$, then $(I - \frac{1}{\lambda_m^{A_{11}}} A_{22})\beta_{II} \geq 0$. Let us assume $\beta_{II} \geq 0$. According to the theorems of Perron and Frobenius, for $\beta_{II} \geq 0$, $(I - \frac{1}{\lambda_m^{A_{11}}} A_{22})\beta_{II} \geq 0$ holds only when the maximum (Perron-Frobenius) eigenvalue of the matrix $(I - \frac{1}{\lambda_m^{A_{11}}} A_{22})$ is positive and consequently the maximum eigenvalue of the matrix $\frac{1}{\lambda_m^{A_{11}}} A_{22}$ is smaller than unit. However, the maximum eigenvalue of the matrix $\frac{1}{\lambda_m^{A_{11}}} A_{22}$ is equal to $\lambda_m^{A_{22}}/\lambda_m^{A_{11}}$ and consequently, because here $\lambda_m^{A_{11}} < \lambda_m^{A_{22}}$, greater than unit and therefore the maximum eigenvalue of the matrix $(I - \frac{1}{\lambda_m^{A_{11}}} A_{22})$ is negative. Consequently, $\beta_{II} \geq 0$ is incongruous and $\beta_{II} \ngeq 0$ holds. So, only for $\beta_{II} \ngeq 0$ is $(I - \frac{1}{\lambda_m^{A_{11}}} A_{22})\beta_{II} \geq 0$.

For the sake of simplification, we shall assume that instead of

$$\beta_{II} \ngeq 0$$

the following holds

$$\beta_{II} < 0,$$

i.e. that not one or some, but all the prices of the non-reproductive commodities are negative.[12]

So, while all the prices of the reproductive commodities are positive, all the prices of the non-reproductive commodities are negative. At the same time, the uniform rate of profit of the overall system is equal to the uniform rate of profit of the reproductive subsystem $(p^{(m)} = p_I^{(m)})$.

Second solution: For $\beta_I = \beta_I^{(m)} = 0$ we get from (18c)

$$x_0 \beta_{II} = A_{22}) \beta_{II} \Rightarrow$$

$$\beta_{II} = \frac{1}{x_0} A_{22}) \beta_{II}.$$

It follows from this relation that

$$x_0 = \lambda_m^{A_{22}} \Rightarrow p = p_{II}^{(m)} \frac{1 - \lambda_m^{A_{22}}}{\lambda_m^{A_{22}}} > 0$$

and

$$\beta_{II} = \beta_{II}^{(m)} > 0.$$

So here, while the prices of all the non-reproductive commodities are positive, the prices of all the reproductive commodities are zero. At the same time, the uniform rate of profit of the overall system is equal to the uniform rate of profit of the non-reproductive subsystem.

12. This supposition simply assumes that the matrices A_{21} and A_{22} are such that $\beta_{II} < 0$. The simpler case, in which $\beta_{II} < 0$, is the case in which only one non-reproductive commodity is produced (which of course, as has been supposed here, enters into its own production).

The former of the above two solutions was first presented by Sraffa (Sraffa (1960), Appendix B), the latter by Vassilakis (Vassilakis (1982))[13]. However, the proper importance has not yet been attached to the co-existence of these two solutions. Evidently, neither of the above two solutions given by (18) in the given *Case 3* is economically significant, i.e. positive, because the first gives negative prices for the non-reproductive commodities, while the second gives zero prices for the reproductive commodities. Each of the two also gives a different uniform rate of profit. The first gives a uniform rate of profit equal to the uniform rate of profit of the reproductive subsystem and the second gives a uniform rate of profit equal to the uniform rate of profit of the non-reproductive subsystem. So, here each of the two solutions gives not only a different uniform rate of profit but *also different relative production prices*. As we shall see further on, the rule applies that the subsystem, the prices of the commodities of which have been assumed to be positive or it follows that they are the only positive prices, determines the uniform rate of profit of the overall system – in the sense that the latter is equal to the uniform rate of profit of that subsystem. This of course holds only in cases where the uniform rates of profit of the above two subsystems are at all events not equal.

Case 4

Here, $A_{22} \geq 0$, $\lambda_m^{A_{11}} = \lambda_m^{A_{22}}$ holds and consequently

$$\lambda_m^A = \max(\lambda_m^{A_{11}}, \lambda_m^{A_{22}}) = \lambda_m^{A_{11}} = \lambda_m^{A_{22}}$$

In the given *Case 4*, the maximum eigenvalue of A recurs once. That is, we have two (equal of course) maximum eigenvalues of A. This is why, as we shall see, we have two solutions of (18), each of which gives the same uniform rate of profit but different relative production prices.

From (18) we get (18a) and (18c) in the given case. We shall give the two solutions of (18a) and (18c) to which we just referred.

First solution: The economically significant solution of (18a) is, as we already know, the following:

$$x_0 = x_0^{(m)} = \lambda_m^A = \lambda_m^{A_{11}} = \lambda_m^{A_{22}} \Rightarrow p = p_I^{(m)} = p_{II}^{(m)} > 0,$$

13. See also Egidi (1975), pp. 11-13.

$$\beta_{I} = \beta_{I}^{(m)} > 0.$$

Given this solution of (18a), (18c) evidently gives, as a consequence of $\lambda_{m}^{A_{11}} = \lambda_{m}^{A_{22}}$,

$$\beta_{II} = \beta_{II}^{(m)} = \text{ indeterminate.}$$

Second solution: For $\beta_{II} = \beta_{II}^{(m)}$ (18c) gives:

$$x_0 = x_0^{(m)} = \lambda_{m}^{A} = \lambda_{m}^{A_{11}} = \lambda_{m}^{A_{22}} \Rightarrow p = p_{I}^{(m)} = p_{II}^{(m)} > 0,$$

i.e., as far as the uniform rate of profit is concerned, the same as the first solution, and

$$\beta_{I} = \beta_{I}^{(m)} = 0.$$

Consequently, the second solution gives the same uniform rate of profit as the first, but a different vector of production prices.

The analytical investigation of the case: $A_{22} \geq 0$ and $\lambda_{m}^{A_{11}} = \lambda_{m}^{A_{22}}$, was conducted first of all by Vouyiouklakis and Mariolis, who showed also the existence of both the above solutions.[14]

Case 5

Here, $A_{22} \geq 0$, $\lambda_{m}^{A_{11}} > \lambda_{m}^{A_{22}}$ holds and consequently

$$\lambda_{m}^{A} = \max(\lambda_{m}^{A_{11}}, \lambda_{m}^{A_{22}}) = \lambda_{m}^{A_{11}}$$

From (18) we get (18a) and (18c) in the given case. Here too there are also two solutions.

First solution: The economically significant solution of (18a) gives, as we already know

$$x_0 = x_0^{(m)} = \lambda_{m}^{A} = \lambda_{m}^{A_{11}} \Rightarrow p = p_{I}^{(m)} > 0,$$

$$\beta_{I} = \beta_{I}^{(m)} > 0.$$

14. See Vouyiouklakis and Mariolis (1993). See also Egidi (1975), pp. 11-13.

Given this solution of (18a), (18c) evidently gives

$$\beta_{II} = \beta_{II}^{(m)} > 0. \quad ^{15}$$

So this solution gives a positive uniform rate of profit, which is equal to the uniform rate of profit of the reproductive subsystem and positive prices for all the –reproductive and non-reproductive– commodities. Here, the uniform rate of profit is determined by (=is equal to) the uniform rate of profit of the reproductive subsystem ($p = p^{(m)} = p_I^{(m)}$), the prices of the commodities of which have been assumed to be positive.

Second solution: For $\beta_I = 0$ we get from (18c)

$$x_0\beta_{II} = A_{22}\beta_{II}.$$

This relation gives

$$x_0 = \frac{1-\lambda_m^{A_{22}}}{\lambda_m^{A_{22}}} \left(> \frac{1-\lambda_m^{A_{11}}}{\lambda_m^{A_{11}}} = \frac{1-\lambda_m^{A}}{\lambda_m^{A}}\right) \Rightarrow$$
$$p = p_{II}^{(m)} (> p_I^{(m)}) > 0$$

and

$$\beta_{II} = \beta_{II}^{(m)} > 0.$$

Here, the uniform rate of profit is equal to the uniform rate of profit of the non-reproductive subsystem, the prices of the commodities of which, it follows, are the only positive prices. At the same time, the prices of the non-reproductive commodities are positive, while those of the reproductive commodities are zero.

The above two solutions differ not only with regard to the uniform rate of profit but also the relative prices. The uniform rate of profit is determined in both solutions as the uniform rate of profit of that subsystem, the prices of the commodities of which are the only positive ones.

15. From (18c) we get (18cc). As a consequence of $\frac{1}{\lambda_m^{A_{11}}} A_{21}\beta_I \geq 0$, $\lambda_m^{A_{11}} > \lambda_m^{A_{22}}$, and indecomposability of A_{22}, it follows from (18cc) that $\beta_{II} = \beta_{II}^{(m)} > 0$.

The existence of the second of the above two solutions –and of course the simultaneous existence of the two solutions– was first shown by Mariolis[16].

At this point, we should like to clarify the following: When in *Cases 3, 4* and *5* we speak of the existence of only two solutions of (18), we do not of course mean that (18) has only two solutions. We simply mean that it has only two "*suboptimal*" economically significant solutions, i.e. only two solutions, each of which gives a positive uniform rate of profit and/or positive prices of reproductive commodities or positive prices of reproductive commodities or positive prices of non-reproductive commodities (*Cases 3* and *4*) or two solutions, of which one is economically significant and the other suboptimal economically significant (*Case 5*).

But as we saw, no solution of (18) is economically significant in *Cases 3* and *4*. Because neither of the two "suboptimal" economically significant solutions given by (18) in *Cases 3* and *4* is economically significant, since neither of them gives positive prices for all the commodities. It is self-evident that the other solutions given by (18) in *Cases 3* and *4* are not economically significant, because self-evidently they can give not only non-positive prices but also a non-positive rate of profit.

As we saw, only in *Case 5* does the first of the two solutions give positive arithmetical values for the uniform rate of profit and for the prices of all the commodities.

Thus, only in *Cases 1, 2* and *5* is there an economically significant solution of (18). In *Cases 3* and *4* there is none. We shall examine the reason for this below.

* * *

Mühlpfort seems to believe that the normalisation of vector β of production prices by means of a normalisation equation, e.g. by means of a normalisation equation like ours (17), is "neutral", i.e. it does not modify the solution given by (18), but simply determines the scalar, apart from which the β given by this solution is fully determined, i.e. it simply changes the relative production prices into absolute production prices without modifying them – and without of course modifying the uniform rate of profit. This today also is almost generally accepted.

16. Mariolis (1993), Ch. IV and Mariolis (1996), pp. 178-181.

However, this is not the situation. In the general case, the solution given by (18) before the normalisation of prices is modified by virtue of the normalisation of prices and in particular by the normalisation commodity used at any time.

Let us see how and why this occurs. Firstly though, we should like to clarify how many different types of normalisation of prices there are. Here, where the system, when decomposable, is split into only two subsystems, the reproductive and the non-reproductive subsystems, and the non-reproductive subsystem is indecomposable, there are, from the point of view that interests us here, only two different normalisations: one, which uses as the normalisation commodity a single or composite commodity, which consists of only reproductive commodities, and the other, which uses as the normalisation commodity a single or composite commodity, which contains at least one non-reproductive commodity.

Thus the normalisation

$$\beta_\alpha = \theta, \ \theta = \text{positive constant,} \tag{17a}$$

where β_α is the price of production of any reproductive commodity α, is a normalisation of the first type, the normalisation commodity of which consists only of reproductive commodities, while the normalisation

$$\beta_\mu = \theta, \ \theta = \text{positive constant,} \tag{17b}$$

where β_μ is the price of production of any non-reproductive commodity μ, is a normalisation of the second type, the normalisation commodity of which is a non-reproductive commodity.

Let us now see what happens with the solution of (18) after the normalisation of prices successively by virtue of (17a) and (17b).

Case 1

In this case, there are evidently only normalisations of the type (17a), i.e. normalisations of which the normalisation commodity consists solely and exclusively of reproductive commodities.

Assuming then that we normalise prices by means of (17a). (17a) entails, because of the indecomposability of A given here, $\beta = \beta^{(m)} > 0$. As we already know, for $\beta = \beta^{(m)} > 0$, (18) gives

$$x_0 = x_0^{(m)} = \lambda_m^A$$

and consequently

$$p = p^{(m)} > 0.$$

Consequently (18) gives the same solution both before and after normalisation.

Case 2

Assuming in this case that we normalise prices by means of (17a). (17a) entails, because of the indecomposability of A_{11}, $\beta_I = \beta_I^{(m)} > 0$.

For $\beta_I = \beta_I^{(m)} > 0$, (18a) gives

$$x_0 = x_0^{(m)} = \lambda_m^{A_{11}} = \lambda_m^{A},$$

i.e. the same solution that it gives before the normalisation of prices.

Assuming that we normalise prices by means of (17b). (17b) entails $\beta_{II} = \beta_{II}^{(m)} > 0$. One can easily ascertain that $\beta_{II} = \beta_{II}^{(m)} > 0$ does not only not conflict with the solution given by (18) before the normalisation of prices, but is a consequence of this solution. So, in the case of normalisation by means of (17b), the solution of (18) remains the same as the solution of (18) before normalisation.

We may therefore conclude that the solution of (18) is not modified by normalisation, i.e. by the normalisation commodity.

Case 3

Assuming in this case that we normalise prices by means of (17a), which, as we already know, entails $\beta_I = \beta_I^{(m)} > 0$. One can easily ascertain that for $\beta_I = \beta_I^{(m)} > 0$, (18) gives the *first* of the two solutions given before the normalisation of prices.

Assuming now that we normalise prices by means of (17b), which, as already know, entails $\beta_{II} = \beta_{II}^{(m)} > 0$. One may easily ascertain that for $\beta_{II} = \beta_{II}^{(m)} > 0$, (18) gives the *second* of the two solutions given before the normalisation of prices.

So, in the given case, the "suboptimal" economically significant solution given by (18) is modified. More specifically, normalisation modifies not only the vector of production prices but also the single rate of profit.

Case 4

Assuming in the given case that we normalise prices by means of (17a), which entails $\beta_{\mathrm{I}} = \beta_{\mathrm{I}}^{(m)} > 0$. One may easily ascertain that for $\beta_{\mathrm{I}} = \beta_{\mathrm{I}}^{(m)} > 0$, (18) gives the first of the two solutions given before the normalisation of prices.

Assuming that we normalise prices by means of (17b), which entails $\beta_{\mathrm{II}} = \beta_{\mathrm{II}}^{(m)} > 0$. For $\beta_{\mathrm{II}} = \beta_{\mathrm{II}}^{(m)} > 0$, (18) gives the second of the two solutions given before the normalisation of prices.

So, in the given case also, normalisation modifies the "suboptimal" economically significant solution given by (18). Only here, each of the two solutions gives different production prices but the same rate of profit.

Case 5

Assuming in the given case that we normalise prices by means of (17a), which entails $\beta_{\mathrm{I}} = \beta_{\mathrm{I}}^{(m)} > 0$. For $\beta_{\mathrm{I}} = \beta_{\mathrm{I}}^{(m)} > 0$, (18) gives the first of the two solutions given before the normalisation of prices.

Assuming that we normalise prices by means of (17b), which entails $\beta_{\mathrm{II}} = \beta_{\mathrm{II}}^{(m)} > 0$. For $\beta_{\mathrm{II}} = \beta_{\mathrm{II}}^{(m)} > 0$, (18) gives not only the first but also the second solution given before the normalisation of prices.

So, here too normalisation modifies the solution given by (18). In the normalisation by means of (17a) we get one solution, while in the normalisation by means of (17b) we get two solutions, one of which coincides with the solution that we get from normalisation by means of (17a). The two different solutions of the three solutions that we get in the case in question for the two types of normalisation differ not only with respect to the uniform rate of profit but also production prices. And most importantly: the first of these two solutions is economically significant, while the second is "suboptimal" economically significant. However, each of the two normalisations gives at least one economically significant solution.

Which solution do we get in each of the above five cases, if we normalise prices by means of (19), i.e. by means of Mühlpfort's normalisation?

In *Case 1* where A is indecomposable, we get the solution that we obtained above both for the normalisation using (17a) and for the normalisation with (17b) – even though here, Mühlpfort's normalisation commodity, i.e. the surplus product U of the given system of production, contains, as a consequence of the indecomposability of A, only reproductive commodities and therefore, Mühlpfort's normalisation (19) is a normalisation of type (17a). This is a consequence of the fact that, because A is indecomposable, the solution is not modified because of normalisation.

Let us now see what happens in the other four cases, in which A is decomposable. When A is decomposable, the normalisation commodity of Mühlpfort's normalisation equation (19), i.e. the surplus product U of the given system of production, necessarily contains at least one non-reproductive commodity. For if it did not contain any non-reproductive commodity, then A would not be decomposable, but indecomposable. So, in those cases in which A is decomposable, Mühlpfort's normalisation (19) is a normalisation of type (17b). Consequently, the solution or solutions that we get in *Cases 2, 3, 4 and 5* if we normalise using (19) is/are exactly the same as that/those (for in *Case 5* we have two solutions) which we get from normalisation using (17b).[17]

All the above appears to be –but is not at all– paradoxical. Because (18) and (17a) or, respectively, (18) and (17b) do not determine the uniform rate of profit and production prices of the originally given system of production [A, z], where z = s, but the uniform rate of profit and the production prices of each normalisation subsystem [A, $u(I-A)^{-1}$], i.e. of the subsystem which produces as its surplus product the respective normalisation commodity u and consequently, as its gross product, $u(I-A)^{-1}$, using the same technique A used

17. In his doctoral thesis, Muhlpfort normalises prices using the normalisation equation

$$s\beta = sa, \tag{19a}$$

where s the gross product of the given system of production and consequently sa the value of that gross product. So here, the gross product functions as a normalisation commodity. When A is indecomposable, this normalisation commodity contains only reproductive commodities and, when A is decomposable, this same normalisation commodity necessarily contains at least one non-reproductive commodity. Therefore, when A is indecomposable, normalisation (19a) is, just as normalisation (19), a normalisation of type (17a) and, when A is decomposable, normalisation (19a) is, just as normalisation (19), a normalisation of type (17b). For this reason, normalisation using (19a) gives exactly the same solutions as those given by normalisation using (19).

also by the originally given system of production or, in the case that A is decomposable, A itself or part of A (here where A, when decomposable, is split into only two parts, the reproductive part of A).[18]

When the normalisation commodity is modified, then the normalisation subsystem $[A, u(I-A)^{-1}]$ is also modified in the sense that it may, if the given technique A is decomposable, use only part of the technique A or the entire technique A.[19] This is tantamount to shifting from one technique to another. This is why $p^{(m)}$ and $\beta^{(m)}$ can be modified with the normalisation commodity u. Because of the existence of this possibility, which is always a given in decomposable techniques in which the non-reproductive commodities enter the production of non-reproductive commodities, the so-called theorem of non-substitution does not hold in decomposable techniques of this type - even though these techniques are linear.[20] For the same reasons, the normalisation commodity modifies the classification of techniques, not only classification based on the w-r-relationship but also classification based on the criterion of minimisation of cost. Because in reality, it is not techniques that are being compared but the corresponding normalisation subsystems. For the same reasons, phenomena of switch and reswitching of techniques appear and disappear because of modifying normalisation.[21]

* * *

18. In the case where A does not contain, as here it contains, also inputs of wage commodities but only inputs of means of production, the normalisation commodity u is not the surplus product of the normalisation subsystem but the *net product* of the normalisation subsystem.
19. If, as we have supposed here for the sake of simplification, A, when decomposable, splits into only two parts, the –self-evidently indecomposable– reproductive part and the –by assumption indecomposable– non-reproductive part, then that part of A which can use the normalisation subsystem is solely and exclusively the reproductive part of A. If however the non-reproductive part of A also splits into two or more parts, then those parts of A which can use the normalisation subsystem are correspondingly more.
20. Regarding the above, see Stamatis (1983), (1988a), Vouyiouklakis/Mariolis (1993), Mariolis (1998a). However, because in these papers A does not contain also inputs of wage commodities but only inputs of means of production, the place of the reproductive and non-reproductive commodities is taken respectively by the basic and non-basic commodities. For the same reason there, the normalisation subsystem is not defined as the subsystem of production that produces the normalisation commodity as its surplus product using only part of the given technique or the entire given technique, but as the subsystem of production which produces the normalisation commodity as its *net product* using only part of the given technique or the entire given technique.
21. See Stamatis (1983), Ch. IV, (1989), (1993), (1998), (1998a), Mariolis (1998), (1998a), (1999a).

It would perhaps not be amiss to refer also to the following result of the preceding analysis. As is known, Bortkiewicz's view that the uniform rate of profit depends solely and exclusively on the technical conditions of production of the reproductive subsystem is generally accepted as correct, as too is the corresponding view of Sraffa that the uniform rate of profit depends solely and exclusively on the technical conditions of production of the basic subsystem. The preceding analysis showed that the uniform rate of profit depends solely and exclusively on the technical conditions of production of the normalisation subsystem, which is why these magnitudes may be modified when the normalisation commodity and consequently the normalisation subsystem is modified. In addition, when the normalisation commodity contains even one non-reproductive or, respectively, non-basic commodity, the rate of profit may be determined solely and exclusively by the technical conditions of production of the non-reproductive or, respectively, of the non-basic commodities, as in *Cases 3, 4* and *5*, where the rate of profit is determined as the rate of profit of the non-reproductive (or, respectively, of the non-basic) subsystem.[22]

* * *

One could construe one of the results of the aforementioned determination of the general rate of profit and the absolute production prices as a refutation of both the neo-Ricardian and the Marxian foundation of profit.

And indeed, we showed that in the case where all the prices are positive and profit is positive, the surplus product is not necessarily positive or semi-positive. This indeed constitutes a refutation of the neo-Ricardian foundation of profit, according to which profit is positive because and only when the surplus product is positive or semi-positive.

We also showed that, although the surplus value is positive, profit is not necessarily positive. Does this constitute a refutation of the Marxian foundation of profit, according to which profit is positive because and only when surplus value is positive? Probably not, for the following reasons: The profit of which we are speaking here is calculated in production prices. As we saw though, "paradoxical", i.e. zero, negative and indeterminate production prices appear in the decomposable systems of *Cases 3, 4 and 5*. We have shown in many of our papers that these paradoxical prices are a consequence of the

22. See Stamatis (1979), Ch. I, (1999a), Mariolis (1996), (1997), (1998a).

arbitrary premise of the existence of a uniform rate of profit[23] and the fact that in cases such as those of the decomposable systems of production of *Cases 3, 4* and *5*, the existence of a uniform rate of profit is, for reasons owing to the technique itself, impossible.

We conclude that it is impossible from the fact that in *Cases 3* and *4*, (18) does not give for any possible normalisation of prices a positive, i.e. economically significant, solution for the rate of profit and production prices, while in *Case 5*, (18) gives only for normalisation (17a) a positive, i.e. economically significant, solution for the above magnitudes, while for normalisation (17b) it gives two solutions, only one of which is positive, i.e. economically significant.

The "invalidity" of the fundamental Marxian theorem which appears here has nothing to do with the content of the said theorem, but is purely and simply a consequence of these "paradoxical" prices and consequently of the non-existence – in the given case – of the possibility of the aforesaid arbitrary premise of the existence of a uniform rate of profit being satisfied.

When this premise can be satisfied for each normalisation of prices, as in *Cases 1* and *2*, then for each normalisation of prices (18) gives $p=p^{(m)}>0$ and $\beta=\beta^{(m)}>0$ and consequently, as follows –given $p^{(m)}>0$, $\beta^{(m)}>0$, $sA>0$– from (16), $U\beta>0$. And because $Ua>0$ always, the fundamental Marxian theorem holds – at least in the case where, as here, production is single and labour homogeneous.

* * *

The inability of the aforementioned premise of the existence of a uniform rate of profit in *Cases 3, 4* and *5* to be satisfied does not only present the so-called fundamental Marxian theorem as being erroneous, even though it is correct, but also has other "paradoxical" consequences.

By way of example, we shall analyse those which appear in *Case 3*. As we already know, if in this case we normalise prices with a normalisation equation of the type (17a), even with (17a) itself, then (18) and (17a) give $\beta_I = \beta_I^{(m)} > 0$,

23. In combination of course with the premise of the existence of uniform production prices and, in the case where A does not contain also inputs of wage commodities, a uniform nominal wage rate.

$\beta_{II} = \beta_{II}^{(m)} < 0$ and $p^{(m)} = p_I^{(m)} > 0$, where $p_I^{(m)}\left(= \dfrac{1-\lambda_m^{A_{11}}}{\lambda_m^{A_{11}}} \right)$ is the uniform rate of profit of the reproductive subsystem.

The prices of all the reproductive commodities are positive, while the prices of all the non-reproductive commodities are negative. The "uniform" rate of profit of the system $p^{(m)}$ is positive.

At the same time, both the nominal capital of the reproductive subsystem K_I

$$K_I = z_I A_{11} \beta_I^{(m)}$$

where z_I, $z_I > 0$, the vector of the gross product of the reproductive subsystem, and the nominal profit of the reproductive subsystem Π_I,

$$\Pi_I = z_I \beta_I^{(m)} - z_I A_{11} \beta_I^{(m)}$$

are positive magnitudes, while both the nominal capital of the non-reproductive subsystem K_{II},

$$K_{II} = z_{II} A_{21} \beta_I^{(m)} + z_{II} A_{22} \beta_{II}^{(m)}$$

where z_{II}, $z_{II} > 0$, the vector of the gross product of the non-reproductive subsystem and $z = \begin{pmatrix} z_I \\ z_{II} \end{pmatrix}$, and the nominal profit of the non-reproductive subsystem Π_{II},

$$\Pi_{II} = z_{II} \beta_{II}^{(m)} - z_{II} A_{12} \beta_I^{(m)} - z_{II} A_{22} \beta_{II}^{(m)}$$

are negative magnitudes. Therefore, the uniform rate of profit of the non-reproductive subsystem $p_{II}^{(m)}$,

$$p_{II}^{(m)} = \frac{\Pi_{II}}{K_{II}} \left(= p_I^{(m)} = p^{(m)} \right)$$

is, as the ratio of two negative magnitudes, *positive*.

Of course one cannot conclude from the above that, for a given uniform and *positive* rate of profit of the overall system, it is possible for there to exist in reality sectors, such as the non-reproductive sectors here, the nominal capital and nominal profit of which are *negative*. The correct conclusion is that here, it is not possible for there to exist a uniform rate of profit for the overall system of production for positive prices of all the commodities and indeed it does not exist. This is for the following reasons: The negativeness of the nominal capital and nominal profit of the non-reproductive subsystem is evidently a consequence of the negativeness of the price of the non-reproductive commodities. But the negative prices that appear here do not mean that it is possible in reality for there to exist –for a uniform and positive rate of profit of the overall system– negative prices of commodities and consequently sectors with negative capital and negative profit; rather they simply mean that here it is not possible for positive prices of all the commodities for there to exist a uniform rate of profit of the overall system. In a strictly logical sense, the negative prices of commodities which appear here constitute the mathematical –not the real economic– conditions for the existence of a uniform rate of profit of the overall system. Consequently, they purely and simply mean the following: Only if it were possible for there to exist negative prices of commodities would it be possible in the given case for there to exist a uniform rate of profit of the overall system; but because in reality it is not possible for there to exist negative prices of commodities and indeed they do not exist[24], it

24. The actual existence of a negative price for a commodity would mean that the seller of the commodity in question would pay the buyer of the said commodity instead of being paid by him – which would of course be irrational, for in such a case why would the seller produce the commodity, since he would incur only loss. The usual interpretation of negative prices, which is given by those who believe that the negative prices that appear in their models have some real economic significance and describe aspects of economic reality, is the following: They maintain that they are prices of products that are not useful but harmful, which prices are paid by the producer of those harmful products in order to be rid of them because this is what he wants or because he is obliged by law to do so. This interpretation is stale. For in their models, production is –without being stated– defined as the production of useful, not harmful things. In the case of single production, it is inconceivable that the one and only product produced by some producer should be not useful but harmful and consequently that he, as a consequence, should not only pay to produce it but also pay to himself destroy it or to assign its destruction to others. In the case of joint production, one could suppose that in certain production processes, harmful by-products are also produced. But if indeed there are such by-products, it is not possible within the framework of models of linear systems of joint production to ascertain which these harmful by-products are. Because in models of linear production systems, the various products as use values differ from one another solely and exclusively with respect to the inputs that are necessary for their production, during the

is not possible in the given case for there to exist a uniform rate of profit of the overall system and it does not exist, as was arbitrarily supposed.

The appearance in the model of negative prices of commodities means: There could only exist a uniform rate of profit of the overall system if it were possible for negative prices of commodities to exist. But because in reality it is not possible for negative prices of commodities to exist and they do not exist, when they do appear in the model it means that in the model it is not possible for a uniform rate of profit of the overall system to exist and it does not exist.[25]

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production processes, into which they enter –if they enter– and with respect to the quantities in which they enter these production processes. Models of linear production systems contain no information about whether a particular product is useful or harmful. Consequently, even if harmful by-products are produced in these production systems, not only do we not know which these are, but –and this is more important– there is no real reason why we should be obliged to associate them with the products whose price is negative. But apart from this, the price of each commodity is modified within the context of the model by virtue of the uniform wage rate (here: the uniform real wage rate), so that a variable wage rate can change it from positive to negative. Would it not be irrational for a commodity, for a certain real wage rate, to have a positive price and consequently to be a useful commodity while for another wage rate, different from the first, to have a negative price and thus –according to the aforesaid interpretation of negative prices– to now become a harmful commodity? What connection is there between the level of the wage rate and the usefulness or harmfulness of a product? Obviously none.

Of course in the case of joint production, harmful by-products can and are produced. Their producer is either obliged (or is not obliged but he himself wants) or is not obliged *and* does not himself want to destroy them. In both these cases, the harmful by-products are not commodities and consequently they do not have a –positive, negative or any other– price. When their producer is obliged (or, without being obliged, himself wants) to destroy them and does indeed destroy them (he himself or some other parties, to whom he has assigned their destruction), then this does not mean that he sells them to some party at a negative price as commodities, but that the cost of their destruction (by him or by others) is borne by him and burdens the cost of the useful commodities, in his production process in which these harmful by-products were produced. When he is not obliged and does not himself want to destroy them and indeed does not destroy them, the problem disappears. So, even if we accept that harmful by-products are produced, the negative prices that appear in models of linear production systems cannot be interpreted as prices of harmful by-products that are to be destroyed.

25. The appearance in the model of the other "irrational" prices, i.e. of zero and indeterminate prices, means exactly the same thing. We shall deal with the case in which zero prices appear below.

We shall show that the negative prices of non-reproductive commodities, which appear in the given *Case 3*, when prices have been normalised using (17a), purely and simply secure satisfaction of the premise of the existence of a uniform rate of profit of the overall system. Let us see how this happens.

In the given case, as we saw previously, (18a) and (18c) hold. Let p_I and p_{II} be the rates of profit of the reproductive and non-reproductive subsystem respectively. For $\beta_I = \beta_I^{(m)} > 0$, which follows from (17a), we get from (18a)

$$p_I = p_I^{(m)} = \frac{1-\lambda_m^{A_{11}}}{\lambda_m^{A_{11}}}$$

For p_{II}, the following evidently holds

$$p_{II} = \frac{z_{II}\beta_{II} - z_{II}A_{22}\beta_{II} - z_{II}A_{21}\beta_I^{(m)}}{z_{II}A_{22}\beta_{II} + z_{II}A_{21}\beta_I^{(m)}}$$

It is evident that for $\beta_{II} > 0$, p_{II} is as bigger (smaller) as the components of $\beta_I^{(m)}$, i.e. the absolute prices of the reproductive commodities, are smaller (bigger). When $\beta_I^{(m)}$ tends to the respective zero vector, then the scalar $z_{II}A_{21}\beta_I^{(m)}$ tends to zero and p_{II} tends to $p_{II}^{(m)}$,

$$p_{II}^{(m)} = \frac{z_{II}\beta_{II} - z_{II}A_{22}\beta_{II}}{z_{II}A_{22}\beta_{II}}$$

$$p_{II}^{(m)} = \frac{1-\lambda_m^{A_{22}}}{\lambda_m^{A_{22}}}$$

At the same time, because of $\beta_I^{(m)} > 0$

$$p_{II} < p_{II}^{(m)}.$$

But because in the given *Case 3* $\lambda_m^{A_{11}} < \lambda_m^{A_{22}}$ holds, we have

$$p_{II} < p_{II}^{(m)} < p_I^{(m)}.$$

Consequently, for positive prices of all the commodities, both reproductive and non-reproductive, it is not possible for a uniform rate of profit to exist, since no matter how small the prices of reproductive commodities in relation to those of non-reproductive commodities, the uniform rate of profit of the non-reproductive subsystem p_{II} always remains smaller than the uniform rate of profit of the reproductive system $p_I^{(m)}$. How is it possible to equalise p_{II} with $p_I^{(m)}$? This can be possible if we allow the prices β_{II} of the non-reproductive commodities to become negative. It becomes immediately clear from the formula for p_{II} which we set out above that, for a given and positive $\beta_I^{(m)}$, p_{II} increases monotonically when β_{II} becomes negative and continuously increases in terms of the absolute amounts of its components. Therefore, for an appropriate negative β_{II}, p_{II} becomes equal to $p_I^{(m)}$, and so we have

$$p_{II} = p_{II}^{(m)} = p_I^{(m)}.$$

i.e. we have a uniform rate of profit of the overall system.

So, the negative prices of the non-reproductive commodities secure in the given case the existence of a "uniform" rate of profit of the overall system, i.e. uniform for negative prices of certain commodities.

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If in the given *Case 3* we normalise prices using a normalisation equation of type (17b), or even with (17b) itself, then (18) and (17b) give $\beta_I = \beta_I^{(m)} = 0$, $\beta_{II}^{(m)} > 0$ and $p = p^{(m)} = p_{II}^{(m)} > 0$.

As one may easily ascertain, for a uniform and *positive* rate of profit of the overall system, because of $\beta_I^{(m)} = 0$ and $\beta_{II}^{(m)} > 0$, the nominal capital and the nominal profit of the reproductive subsystem are zero magnitudes, while the nominal capital and nominal profit of the non-reproductive subsystem are positive magnitudes.

One cannot conclude however from this that for a uniform and *positive* rate of profit of the overall system it is possible in reality for there to exist sectors, such as the reproductive sectors here, whose nominal capital and nominal profit are zero, as emerges in the model here. The correct conclusion is that in the given case, it is not possible for a uniform rate of profit of the overall system to exist for positive prices of all the commodities and indeed it does not exist. This is for the following reasons: Zero prices of commodities do not exist in reality.[26] The zero prices of reproductive commodities which appear here are nothing more than mathematical – not real economic – conditions for the existence of a uniform rate of profit of the overall system. In a strictly logical sense they mean: Only if it were possible for there to exist zero prices of commodities would it be possible in the given case for there to exist a uniform rate of profit of the overall system; but because in reality it is not possible for there to exist zero prices of commodities, there is no uniform rate of profit of the overall system in the given case.

We shall show below that the zero prices of reproductive commodities, which appear in the given *Case 3*, when prices have been normalised using (17b) (or with any other normalisation equation of the type (17b)), is nothing more than the mathematical -mathematical and not economic!- condition for the existence of a uniform rate of profit of the overall system, i.e. a purely mathematical condition, under which the axiomatically set –but in the given case economically, i.e. for the positive prices of all the commodities, impossible to be fulfilled and indeed unfulfilled– premise of the existence of a uniform rate of profit of the overall system.

According to that set out above, in the given case (18a), (18c), (17b) and $\lambda_m^{A_{11}} < \lambda_m^{A_{22}}$ hold. As a consequence of (17b), $\beta_m^{(II)} > 0$.

Let $p_I^{(m)}$ and $p_{II}^{(m)}$ be the rates of profit of the reproductive and non-reproductive subsystems respectively. Thus, (18a) and (18c) respectively take the forms

$$\frac{1}{1+p_I^{(m)}}\beta_I = A_{11}\beta_I \tag{α}$$

and

$$\frac{1}{1+p_{II}^{(m)}}\beta_{II} = A_{12}\beta_I + A_{22}\beta_{II}. \tag{b}$$

For positive prices of reproductive commodities, i.e. for $\beta_I > 0$, we get from (a)

26. If they did exist, it would mean that the producers of commodities who produce commodities with zero prices do not sell but rather always give away these commodities.

$$p_{I}^{(m)} = \frac{1-\lambda_{m}^{A_{11}}}{\lambda_{m}^{A_{11}}}.$$

From (b) it is immediately clear that $p_{II}^{(m)}$ is as bigger (smaller) as the components of β_I, i.e. the absolute prices of the reproductive commodities, are smaller (bigger). Consequently $p_{II}^{(m)}$ reaches its maximum arithmetical value when $\beta_I = 0$. When $\beta_I = 0$, we get from (b) for $p_{II}^{(m)}$:

$$p_{II}^{(m)} = \frac{1-\lambda_{m}^{A_{22}}}{\lambda_{m}^{A_{22}}}.$$

Because of $\lambda_{m}^{A_{11}} < \lambda_{m}^{A_{22}}$ which holds here, the following evidently holds

$$p_{I}^{(m)} = \frac{1-\lambda_{m}^{A_{11}}}{\lambda_{m}^{A_{11}}} > p_{II}^{(m)} = \frac{1-\lambda_{m}^{A_{22}}}{\lambda_{m}^{A_{22}}}.$$

Consequently it is not possible for a single rate of profit $p^{(m)}$ to exist for the overall system of production,

$$p^{(m)} = p_{I}^{(m)} = p_{II}^{(m)},$$

with positive prices for *all* the commodities, because for positive prices of the reproductive commodities ($\beta_I > 0$), the rate of profit of the reproductive subsystem $p_{I}^{(m)}$ is always greater than the *maximum possible* rate of profit of the non-reproductive subsystem $p_{II}^{(m)}$, which arises for positive prices of the non-reproductive commodities ($\beta_{II} > 0$) and for zero prices of the reproductive commodities ($\beta_I = 0$), and consequently always greater than any other rate of profit of the non-reproductive subsystem, which arises for positive prices of the non-reproductive commodities ($\beta_{II} > 0$) and for *positive* prices of the reproductive commodities ($\beta_I > 0$).

Despite this however, these two unequal rates of profit $p_{I}^{(m)}$ and $p_{II}^{(m)}$ can mathematically be equalised, so that there exists a uniform rate of profit of the overall system $p^{(m)}$,

$$p^{(m)} = p_{I}^{(m)} = p_{II}^{(m)}.$$

Let us see how this is possible. For profit Π_I, capital K_I and the rate of profit $p_I^{(m)}$ of the reproductive subsystem, the following hold respectively:

$$\Pi_I = z_I \beta_I^{(m)} - z_I A_{11} \beta_I^{(m)},$$

$$K = z_I A_{11} \beta_I^{(m)}$$

and

$$p_I^{(m)} = \frac{z_I \beta_I^{(m)} - z_I A_{11} \beta_I^{(m)}}{z_I A_{11} \beta_I^{(m)}}.$$

For $\beta_I^{(m)} = 0$, we get from this last relation

$$p_I^{(m)} = \frac{0}{0} = \text{indeterminate}$$

Also for $\beta_I^{(m)} = 0$, we get from (b):

$$p_{II}^{(m)} = \frac{1 - \lambda_m^{A_{22}}}{\lambda_m^{A_{22}}}.$$

The magnitude $p_I^{(m)}$, being an indeterminate magnitude, may be set equal to any magnitude. Therefore we may set

$$p_I^{(m)} = p_{II}^{(m)} = \frac{1 - \lambda_m^{A_{22}}}{\lambda_m^{A_{22}}} = (p^{(m)}).$$

So, the zero prices of reproductive commodities ($\beta_I = 0$), which appear in the given *Case 3* when prices have been normalised using a normalisation of the type (17b) and consequently $\beta_{II} > 0$, lack any economic significance and consequently do not mean that zero prices of commodities can and do exist, but are purely and simply a mathematical –not economic!– condition for the existence of the axiomatically supposed –but in the given case not possible to exist for technical/economic reasons– uniform rate of profit of the overall system of production.

* * *

As we saw, in the given *Case 3*, as well as when one normalises prices –either with (17a) or (17b), which constitute all the possible types of price normalisation here– a uniform rate of profit of the overall system of production cannot exist for the positive prices of *all* the commodities.

The at times negative and at times zero prices of commodities which appear in the given *Case 3* for each possible normalisation of prices, as well the consequence of these negative and zero prices, the existence of sectors with negative capital and negative profit for a "uniform" and positive rate of profit of the overall system of production or, correspondingly, of sectors with zero capital and zero profit, are not real economic phenomena, but purely and simply mean that in the given *Case 3* it is not possible –for positive prices of commodities– for a uniform rate of profit of the overall system of production to exist.

So, *just as* these "irrational" consequences of the "irrational" prices which appear in the model do not describe real economic phenomena but mean, when they appear, that it is not possible for a uniform rate of profit of the overall system of production to exist for the positive prices of all the commodities, *so too* the apparent invalidity –as a consequence of the appearance of "irrational" prices– of the so-called fundamental Marxian theorem does not mean that this theorem is not correct, but, whenever this invalidity appears, that it is not possible –for the positive prices of all the commodities– for a uniform rate of profit of the overall system of production to exist.

* * *

Mühlpfort considers that the system of equations comprising (18) and (19) had a solution and moreover one and only one solution. He further considers it to be self-evident that this solution is of economic significance, i.e. that it gives a positive arithmetical value for the uniform rate of profit and positive arithmetical values for production prices. In reality, Mühlpfort does not solve the system of equations for determining the uniform rate of profit and production prices (as, for example, Bortkiewicz solves it). But this is virtually of no importance. For the solution of the system is no longer an economic but a purely mathematical problem. What is important is that Mühlpfort was the first to correctly set out the problem and that in doing so, he fully formulates a system of inputs-outputs, which he furthermore presents in a very modern way with equally modern symbolism.[27]

27. As we saw, in order to determine the coefficients of deviation of the production prices from the respective values and the uniform rate of profit, Mühlpfort introduces a normalisation equation of relative production prices and concludes (in his thesis) that "then we have n+1 equations for the n+1 unknowns x_0, x_1, x_2, ..., x_n. In this way, these unknowns are determined..." or (in his article) "We thus found n + 1 equations, in which we ascertain n+1 unknowns, i.e. x_1, x_2, ..., x_n and x_0". It appears therefore that Mühlpfort leaves aside the following questions: a) Does the said system of n+1 equations with the n+1 unknowns have a solution? b) If yes, is that solution unique? c) If it is unique, is it also economically significant, i.e. positive? And, if there is more than one solution, is there a positive one among them? It is clear that Mühlpfort could not answer the second and third of the above three questions, for the respective answers are based on the relevant theorems of Perron-Frobenius for non-negative matrices which were formulated during the period 1907-1912 and on the conditions set out much later by Hawkins-Simon. On the other hand however, the answer to the first of the above three questions *should have been* known to Mühlpfort. To be more precise, it should have been known to him that the homogeneous system of equations

$$(x_0I-A)\beta=0 \tag{18}$$

at the same time determines the vector β of relative production prices (which constitutes the right eigenvectors of A) and x_0 (which constitutes the eigenvalues of A). Consequently, it *should have been* known to him that this homogeneous system of n equations with n+1 unknowns always has a solution (or solutions). We surmise that these general mathematical properties of the system (18) *should have been* known to Mühlpfort, who, as he states in his Vita, which is contained in his thesis, he also attended lectures in "rebus physics" at the University of Berlin. Because in his time, the said systems of equations had not only been investigated, but had also been used for solving specific problems. Thus, for example, it is known that in 1840, Urbain Leverrier (1811-1877) had, within the framework of the problem of calculating the position of the planet Neptune on the basis of the empirically observed anomalies of the orbit of Uranus, developed a particularly smart method for calculating the coefficients of the characteristic polynomial of a square matrix (I owe this piece of information to Theodore Mariolis). However, we cannot say whether Mühlpfort had clear knowledge of the possibility of determining the production prices (with the exception of one or more scalars) and the uniform rate of profit *before* introducing a normalisation equation (which he introduces afterwards, in order to calculate the absolute production prices) or whether he believed that only *after* the introduction of a normalisation equation was it possible (because then the multitude of equations becomes equal to the multitude of unknowns in the system) to determine *all* the unknowns of the system. However, because (a) he refers only to the determination of the uniform rate of profit *after* the introduction of a normalisation equation, (b) he proceeds –both in his thesis and in his article– to measure the equations and unknowns of the system *only after* the introduction of a normalisation

The reader who is familiar with the subject-matter of the conversion of labour values into production prices will no doubt have noted that Mühlpfort does not directly determine the rate of profit p, but first the magnitude $x_0\,(=1:(1+p))$ and then the rate of profit. And also that he does not directly determine production prices β, but first the ratios x of production prices to labour values and then –for given values– production prices β. This method of indirectly determining production prices, which is used also by von Bortkiewicz, must therefore come from Mühlpfort.

In view of all this, it emerges that the first correct formulation/solution to the so-called "transformation problem" was not given by Dmitriev and Bortkiewicz, but by Mühlpfort in 1895.[28]

We Marxist economists for a long time believed that the so-called transformation problem consisted in the quantitative determination of production prices by means of labour values, where we considered labour values to be the quantities of supposed homogeneous labour embodied in the corresponding commodities. At the same time, we supposed, without further investigation, that both the production prices, i.e. the prices that emerge for a uniform wage rate and a uniform rate of profit, and labour values existed and were positive and uniquely determined magnitudes. The transformation

equation and (c) he makes no mention whatsoever of the possibility of the said system of n+1 equations and n+1 unknowns being indeterminate or incompatible, we surmise that the second case is the most likely, i.e. that Mühlpfort did not realise that it was possible on the basis of (18) to determine the vector of production prices (with the exception of one or more scalars) and the uniform rate of profit.

28. To be precise, Mühlpfort had solved the transformation problem by as early as 1893. The solution that he presents in his article, on which we have commented here, is contained in his doctoral thesis which was published in 1893. The only difference lies in the fact that Mühlpfort normalises prices in his thesis by setting the sum of prices equal to the sum of labour values, while in his article, he sets total profit equal to total surplus value. See Wolfgang Muehlpfordt, *Preis und Einkommen in der privatkapitalistischen Gesellschaft. Inaugural-Dissertation zur Erlangung der Doktorwürde von der philosophischen Fakultät der Albertus-Universität zu Königsberg i. Pr., genehmigt und am Freitag den 22 Dezember 1893, mittags 12 Uhr, mit den beigefügten Thesen öffentlich verteidigt, Königsberg, Hartungsche Buchdruckerei, pp. 20-28.* (From Mühlpfort's reference to the above thesis as being his own thesis, it follows that the author of the article Dr. Mühlpfort and the author of the thesis Wolfgang Muehlpfordt are one and the same person and that the name "Mühlpfort" instead of "Muehlpfordt" in the article is obviously due to the transcription of "ue" to "ü" and the omission of the "d".).

problem as meant in this sense cannot have and does not have any meaning or solution. The reason for this is the following: In the general case, labour values are always positive but not always uniquely determined, while production prices are not always positive nor uniquely determined. Consequently, it is not possible for unambiguous correspondence to exist between labour values and production prices and it does not exist, and thus one cannot speak in terms of the former determining the latter.

Even when values are, as in the case of single production, to which we restricted ourselves here, uniquely determined and positive, production prices are not only not –either before or after their normalisation– always uniquely determined, but not even always positive, since certain of them may be zero, negative or indeterminate. And things get even worse in the case of joint production (which constitutes the rule), where not only are production prices not always positive and uniquely determined, but labour values are, on the one hand, always positive but, on the other, not uniquely determined (see also Stamatis (1983a)).

So for two important reasons, labour values cannot determine uniquely determined *positive* production prices:
Firstly, because in the general case (i.e. including also the case of joint production) values are positive but not always uniquely determined magnitudes and,
Secondly, because in the general case (i.e. including the case of joint production, but also that of decomposable systems of production) they cannot always be defined and consequently there are not always production prices.[29]

But even if both labour values and production prices were always positive and uniquely determined magnitudes – even then it would not be possible for

29. Production prices are, as is known, defined as those prices which apply when the rate of profit is uniform. So, the cases in which it is not possible for there to be a uniform rate of profit and in which, consequently, production prices are not uniquely determined and positive, evidently mean that production prices cannot be defined and consequently do not exist in the general case.
We showed previously that in single production, "irrational" production prices appear only in decomposable techniques. We should like to observe at this point that in joint production, "irrational" production prices appear also in partly productive or, correspondingly, partly surplus productive *indecomposable* techniques. We call a technique partly productive (or partly surplus productive) when it cannot produce all the possible positive or semi-positive net products (surplus products), but only one or some of them.

labour values to determine production prices. For, on the one hand, the former are determined exclusively by the given production technique $[\bar{A},\ell]$, while, on the other, the latter by the given production technique $[\bar{A},\ell]$ and the given real or nominal wage rate (here: by the given real wage rate d).

So, the solution to the transformation problem cannot consist in showing the existence of a relationship for determining the production prices by means of labour values. This is so because, as we have shown, not only is such a relationship non-existent, but chiefly because the real problem lies elsewhere. The real problem that we are called upon to solve is the following: How, through the exchange of the products of the various forms of concrete private useful labours for money and their transformation into capitalist commodities, firstly, are the various concrete forms of private useful labours embodied in those commodities transformed into abstract social, in general useful labour, secondly, does the quantity of abstract social, in general useful labour embodied in each commodity, i.e. its true value, take the form of the real market price in money terms (not the form of the price of production!) of that commodity and thus, thirdly, does the abstract social, in general useful labour take the form of real money (regarding these issues: Stamatis (1995a), pp. 15-61, Stamatis (1998b), pp. 49-71 and 132-143, Mariolis (1998b), pp. 73-80, Mariolis (1999), (2000)).

These issues cannot be treated solely and exclusively within the framework of models like the above, i.e. within the framework of the model of a linear production system. The reason is that fictitious money, which we introduce to the model in the only way that we can introduce it, i.e. through an equation for the normalisation of prices, as we have already shown, is not neutral, since in the general case the rate of profit and prices are modified by virtue of the normalisation commodity.

So, within the framework of this model, the rate of profit and prices cannot be regarded as isomorphous representations of the real profit rate and of real prices. But even if we decide to keep the normalisation of prices invariable, with the justification that our intention is to deal not with the rate of profit and the prices of the given system of production, but with the rate of profit and the prices of the normalisation subsystem, which corresponds to the given and by assumption invariable normalisation, we would come up against difficulties. The prices of each normalisation subsystem are always positive or semi-positive (see Stamatis (1988a)). More specifically, in the case where the given normalisation subsystem is decomposable, it is possible for zero prices of all the reproductive commodities to appear – which (zero prices) cannot be regarded as isomorphous representations of the real prices of the reproductive commodities.

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