POLITICAL ECONOMY - **6** - SPRING 2000 - p.p. 161 - 166

A Generalization of the Theorem on the Eigenvalues of Positive or Semipositive Similar Matrices and Its Application in the Economic Theory

by
Georg Stamatis

We give in the following a generalization of the well-know theorem on the eigenvalues of two different square and of the same order positive or semipositive matrices, so that it holds in a broad sense for two square positive or semipositive matrices of different order. Afterwards we give an application of this generalized theorem in the economic theory.

Let A, $A \geq 0$, be a nxn positive or semipositive matrix and let B, $B \geq 0$, be a kxk positive or semipositive matrix, where $A \neq B$, $n \geq 1$, $k \geq 1$ and $n \gtreqless k$. The matrices A and B are either in common strict sense or in a broad sense similar matrices and they have min(n,k) common eigenvalues, including the common maximum eigenvalue, if there exist matrices H and Z, where H is a kxn matrix and Z is a nxk matrix, so that

$$H \geq 0, \tag{1}$$

$$ZH = I, \tag{2}$$

$$HAZ = B \tag{3}$$

and hence, given (2),

$$HAZ = B \Rightarrow$$

$$ZHAZH = ZBH \Rightarrow$$

$$A = ZBH \tag{4}$$

or / and

$$Z \geq 0, \tag{1a}$$

$$HZ = I, \tag{2a}$$

$$ZAH = B$$

and hence, given (2a)

$$\begin{aligned} &ZAH=B \Rightarrow \\ &HZAHZ=HBZ \Rightarrow \\ &A=HBZ. \end{aligned} \tag{4a}$$

Proof:

Suppose that (1), (2) and (4) hold.
It holds

$$\lambda^B\psi = \psi HAZ \tag{5}$$

where ψ are the k left-hand eigenvectors of matrix HAZ(=B). For one of these eigenvectors, namely for the eigenvector $\psi_m H$ that corresponds to the maximum eigenvalue λ_m^B of HAZ, hold

$$\lambda_m^B \psi_m = \psi_m HAZ \tag{6}$$

and, because of $HAZ \geq 0$,

$$\psi_m \geq 0. \tag{7}$$

Postmultiplying both sides of (5) and (6) by H and taking into account (2), we get respectively

$$\begin{aligned} &\lambda^B\psi H=\psi HAZH \Rightarrow \\ &\lambda^B\psi H=\psi HA \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\lambda_m^B \psi_m H = \psi_m HAZH \Rightarrow \\ &\lambda_m^B \psi_m H = \psi_m HA \end{aligned} \tag{9}$$

According to (8), ψH are left-hand eigenvectors of the nxn matrix A that correspond to those eigenvalues λ^B of the kxk matrix B that are equal to eigenvalues λ^A of A.

The number of these common eigenvalues of A and B is, when $n \leq k$, equal to n and, when $n \geq k$, equal to k. Consequently this number is equal to min(n,k).

To this number min(n,k) of the common eigenvalues of A and B belongs for the following reasons the common maximum eigenvalue of A and B: When n=k, then according to (2) Z is the inverse of H and according to (3) A and B are in common strict sense similar matrices[1]. And when $n \neq k$, then according to (1) and (7) we have

$$\psi_m H \geq 0. \tag{10}$$

Thus $\psi_m H$ is the one left-hand eigenvector of A that corresponds to the maximum eigenvalue λ_m^A of A, so that

$$\lambda_m^A \psi_m H = \psi_m H A. \tag{11}$$

From (9) and (11) we obtain

$$\lambda_m^B = \lambda_m^A \tag{12}$$

Consequently when $n \neq k$, then the matrices A and B are in a broad sense similar matrices[2], since they have min(n,k) common eigenvalues including their common maximum eigenvalue.

Suppose now that, instead of (1), (2) and (3), (1a), (2a) and (3a) holds. Then the proof can be given in accordance to the above.

Q.E.D.

The above generalized theorem plays a rôle in the economic theory.

Let D, $D \geq 0$, be a nxk matrix. And let Ω, $\Omega \geq 0$, be a kxn matrix, where $n \geq 1$, $k \geq 1$ and $n \gtrless k$. Matrix D is the matrix of the k different real wages rates, the column D_j, $D_j \geq 0$, of which denotes the real wages paid for a unit of labour of the type j, j=1,2,...,k. Matrix Ω is the matrix of the labour values of the n produced commodities, the column Ω_i, $\Omega_i \geq 0$, of which denotes the quantities of the k types of direct and indirect labour, that requires the production of a unit of the commodity i, i=1,2,...,n.

1. In this case, where obviously $H=Z^{-1}$ and $Z=H^{-1}$, is either $H>0$ and consequently Z contains also negative elements or $Z>0$ and consequently H contains also negative elements or both H and Z are semipositive. When both H and Z are semipositive, then every row and every column of both H and Z contains only one elements positive (See J. Magnan de Bornier, Production jointe et modèle temporel, in: Ch. Bidard (ed.), *La production jointe*, Economica, Paris 1984, pp. 98-110). In the same case hold not only (2) and (3) but also (2a) and (3a), and if both H and Z are semipositive holds, in addition, not only (1) but also (1a).
2. In the trivial case in which A and B are identity matrices of different order we get from (3) that HZ=I, i.e., that also (2a) and consequently (3a) hold. Hence in this case hold not only (2) and (3) but also (2a) and (3a).

According to the above the nxn matrix $D\Omega$, $D\Omega \geq 0$, and the kxk matrix ΩD, $\Omega D \geq 0$, are either in common strict sense or in the above broad sense similar matrices.

Proof:

The matrices $D\Omega$ and ΩD are similar matrices if there exist matrices H and Z, so that

$$H \geq 0, \tag{1aa}$$

$$ZH=I \tag{2aa}$$

and

$$H(D\Omega)Z=\Omega D \tag{3aa}$$

or / and

$$Z \geq 0, \tag{1aaa}$$

$$HZ=I \tag{2aaa}$$

and

$$Z(\Omega D)H=D\Omega. \tag{3aaa}$$

Provided that matrix ΩD is non-singular there exist matrices H and Z,

$$H=D$$

and

$$Z=(\Omega D)^{-1}\Omega,$$

they satisfy (1aa), (2aa) and (3aa). Hence the matrices $D\Omega$ and ΩD are similar matrices.[3] Consequently it holds

3. Since ex hypothesi $\Omega D \geq 0$, matrix $(\Omega D)^{-1}$ *can* contain negative elements. Matrix $(\Omega D)^{-1}$ contains negative elements if $\Omega D>0$ and it does not contain negative elements if ΩD is semipositive (see footnote 1). Consequently, since ex hypothesi $\Omega \geq 0$, matrix $Z[=(\Omega D)^{-1}\Omega]$ can contain negative elements only if $\Omega D>0$ and it does not contain negative elements if ΩD is semipositive.

It is ex hypothesi $D(=H) \geq 0$. So, provided that D and Ω are non-singular matrices, when n=k and consequently matrix Z is the inverse of matrix H,

$$Z=(\Omega D)^{-1}\Omega=D^{-1}\Omega^{-1}\Omega=D^{-1}(=H^{-1}),$$

then matrix Z contains negative elements if $H(=D)>0$ and it does not contain negative elements but is semipositive if matrix $H(=D)$ is semipositive (see footnote 1). When n=k,

$$\lambda_m^{D\Omega} = \lambda_m^{\Omega D}, \tag{12}$$

where $\lambda_m^{D\Omega}$ is the maximum eigenvalue of $D\Omega$ and $\lambda_m^{\Omega D}$ is the maximum eigenvalue of ΩD.

It can also be proved[4] that in the case that the given production techniques is linear, irreducible[5] and productive, profits, reckoned in production prices, are positive, only if

$$(0<)\lambda_m^{D\Omega} < 1 \tag{13}$$

and, because of (12), only if

$$(0<)\lambda_m^{\Omega D} < 1, \tag{14}$$

and surplus value is positive or semipositive (surplus value is a $k \times 1$ vector), only if

$$\Omega D s \leq s, \; s = (1,1,\ldots,1)_k^T \tag{15}$$

So, when (13) and consequently (14) hold, and hence profits are positive, obviously condition (15) doesn't necessarily hold, i.e. it is possible that the vector of surplus value contains -except of positive or semipositive and equal to zero- negative components too.

Therefore it seems as if the marxian assertion, according to which profits are positive because surplus value is positive (or semipositive), is incorrect.

---

then obviously hold not only (1aa), (2aa) and (3aa) but also (1aaa), (2aaa) and (3aaa). And when the matrices $D\Omega$ and ΩD are identity matrices of different order ($n \neq k$), then obviously hold not only (2aa) and (3aa) but also (2aaa) and (3aaa) (see footnotes 1 and 2).

4. See for the following Georg Stamatis: Marxian versus neoricardian profit theory, *Indian Economic Journal*, Vol. 46, April-June 1998-99, No 4, pp. 74-80.
5. Or it is reducible, but of the type, according to which the non-basics don't enter the production of non-basics. This presupposition excludes the possibility of the appearance of non-positive (i.e. zero, negative, indeterminate and tending to infinity) and hence economically meaningless production prices, which can appear in reducible techniques even if (13) and consequently (14) hold. Those non-positive production prices are in fact *mathematical conditions* that ensure the axiomatically presupposed existence of a uniform profit rate in reducible techniques for which (13) and consequently (14) hold.

But in the case, in which $k=1$ and consequently the labour is homogeneous, (15) obviously always holds when (13) and consequently (14) hold. This case shows that the Marxian thesis would always (i.e. even if $k>1$) be correct, provided that it was completed with a theory that reduces the k inhomogeneous types of labour to one type of homogeneous labour. Thus, the crux of the issue is the conception, explanation and presentation of the *real process* which transforms concrete and hence inhomogeneous labours to abstract and hence homogenous labour.