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# A note on the Normalizations that imply Linear w-r Relations in Decomposable Single Production Techniques

*by*

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In their recent published book, *Prices, Profits and Rythms of Accumulation*, Abraham-Frois and Berrebi describe cases where multiple normalization commodities imply a linear w-r relation in decomposable linear production techniques. References to the literature are absent, consequently we may conclude that this is an original contribution to the analysis by Abraham-Frois and Berrebi. However, the case of multiple normalization commodities that imply a linear w-r relation has been found and described thoroughly by Vassilakis in a brief note written in 1982 and circulated privately, translated in German, published in *Hefte für Politische Ökonomie* (Heft 5, 1983) and published again as an Appendix of the book *Sraffa und sein Verhältnis zu Ricardo und Marx* (G. Stamatis, 1984).

Surprisingly, the article of Vassilakis remain till today almost unknown, although it has the mathematical precision and economical reasoning of an original work. In the following we are going to recall briefly the main arguments of Vassilakis note and attempt to evaluate the recent contribution of Abraham-Frois and Berrebi in comparison to the article of Vassilakis.

After a brief introduction to the subject Vassilakis considers a linear production technique, producing n commodities, m basics and (n–m) non-basics. The technique in the article of Vassilakis is described by the nxn input matrix A

$$A=\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where $A_{11}$ is a mxm indecomposable matrix, $A_{12}$ is a semipositive mx(n–m) matrix and $A_{22}$ is a (n–m)x(n–m) *indecomposable* matrix. We will return to the latter assumption later. Additionaly Vassilakis assumes that $\lambda^{A}_{max} < 1$, where

$\lambda^{A}_{max}$ denotes the Perron-Frobenius or the maximum eigenvalue of matrix A. The equations that define the standard commodity are according to Vassilakis:

$$(1 + R)Aq = q, \quad q \geq 0, \quad \text{or}$$

If q is partioned to a mx1 and a (n–m)x1 vector

$$(1+R)\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \geq 0, \tag{1}$$

where $q = (q_1, q_2)^T$ is the standard commodity. As it is well known Sraffa excludes by definition the case $q_2 \neq 0$, and assumes $q_2 = 0$. This assumption defines the Sraffian standard commodity by means of the equation:

$$(1 + R_1)A_{11}q_1 = q_1, \tag{2}$$

where $R_1 = \frac{1}{\lambda^{A_{11}}_{max}} - 1$ and $\lambda^{A_{11}}_{max}$ is the Perron-Frobenius, or the maximum eigenvalue of $A_{11}$.

Vassilakis, however, observes that if $q_2 \geq 0$, then

$$(1 + R_2)(A_{11}q_1 + A_{12}q_2) = q_1 \tag{3}$$

$$(1 + R_2)A_{22}q_2 = q_2 \tag{4}$$

From equation (4) we have $R_2 = \frac{1}{\lambda^{A_{22}}_{max}} - 1$, $q_2 > 0$. From equation (3) it follows that:

$$q_1 = (I_1 - (1 + R_2)A_{11})^{-1}(1 + R_2)A_{12}q_2, \tag{5}$$

and by a well known corollary of Perron-Frobenius Theorems $(I_1 - (1 + R_2)A_{11})^{-1}$ is positive if, and only if, $1 + R_2 < 1 + R_1$, i.e. $R_2 < R_1$ or $\lambda^{A_{11}}_{max} < \lambda^{A_{22}}_{max}$. Vassilakis concludes that "... the standard commodity may contain all the non-basic commodities in positive proportions too, provided that $\lambda^{A_{22}}_{max} < \lambda^{A_{11}}_{max}$ ". (Vassilakis, 1984, p. 444).

A corollary of the above analysis of Vassilakis is that only two standard commodities exist under the assumption of the indecomposability of $A_{22}$. Both standard commodities imply a linear w–r relation, as it can be easily shown. Let p denotes the 1xn vector of production prices, r denotes the uniform profit rate and w denotes the uniform wage rate. The equations that determine production prices, profit rate and wage rate are:

$$pA(1+r) + w\ell = p\,, \quad \text{or} \tag{6}$$

$$p_1A_{11}(1+r) + w\ell_1 = p_1\,, \quad \text{and}$$

$$(p_2A_{22} + p_1A_{12})\,(1+r) + w\ell_2 = p_2\,.$$

If one normalizes prices using sraffian standard commodity, obtains the following w–r relation:

$$[(pA(1+r)+w\ell]\begin{bmatrix} q_1 \\ 0 \end{bmatrix} = p\begin{bmatrix} q_1 \\ 0 \end{bmatrix}, \quad \text{or} \tag{7}$$

after well known manipulations and using the normalization identities $pq = p_1q_1 = 1$ and $\ell q = \ell_1 q_1 = 1$ we get:

$$w = \frac{R_1 - r}{1 + R_1} \tag{8}$$

Similarly, if one normalizes prices using the standard commodity of Vassilakis, obtains the following w–r relation:

$$[(p_1,p_2)\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}(1+r) + w(\ell_1,\ell_2)]\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = (p_1,p_2)\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \tag{9}$$

where 1Xn vectors p and $\ell$ are partitioned in $(p_1, p_2)$ and $(\ell_1, \ell_2)$ respectively. Obviously $p_1$ and $\ell_1$ are mX1 vectors and $p_2$ and $\ell_2$ are $p_1$ and (n–m)X1 vectors.

One can re-write (9) as following:

$$[p_1A_{11}q_1 + p_1A_{12}q_2 + p_2A_{22}q_2]\,(1+r) + w(\ell_1q_1 + \ell_2q_2) = p_1q_1 + p_2q_2 \tag{10}$$

After manipulations of the equation (10) using the presupposed normalization equations $pq = p_1q_1 + p_2q_2 = 1$ and $\ell q = \ell_1q_1 + \ell_2q_2 = 1$ we get:

$$w = \frac{R_2 - r}{1 + R_2} \tag{11}$$

Thus using the standard commodity of Vassilakis we get a second linear w–r relation different from the linear relation implied by sraffian standard commodity. The existence of the Vassilakian standard commodity presupposes, however, the relation $R_2 < R_1$ or $\lambda_{max}^{A_{11}} < \lambda_{max}^{A_{22}}$ . Otherwise $q_2 = 0$, even if the equality between $R_2$ and $R_1$ holds (or $\lambda_{max}^{A_{11}} = \lambda_{max}^{A_{22}}$ ), and the standard commodity of Vassilakis does not exist. This analysis of the multiple w–r relations does not exist in an explicit form in the brief article of Vassilakis, but we must recall that this article has no heuristic purposes and it is devoted to the proof of existence of standard commodities that include non basics.

Finally, Vassilakis conclude from his analysis that the maximum profit rate of a decomposable technique of linear single production is the minimum of the maximum profit rates of the basic and the non-basic sub-techniques. The latter is called by Vassilakis *physical own rate of reproduction of non-basic commodities*, which is the most appropriate term to describe it. On the other hand Abraham-Frois and Berrebi call this magnitude *maximum profit rate of the blocking goods* which is misleading, because one can conclude that these goods or commodities act as a barrier and prevent profit rate from attaining its maximum value, which is the maximum profit rate of the basic commodities. However, this is not the case, because a normalization à la Vassilakis imply a *different* w–r linear relation and a different maximum profit rate, which is the one of the non-basic commodities.

At this point we recall the analysis by Abraham-Frois and Berrebi of multiple linear w-r relations in decomposable single production techniques. This analysis of decomposable techniques of single production takes place in Chapter 3 of their book, which has the title: *Irregular and decomposable systems.* A further analysis of decomposable single production techniques takes place in Chapter 5 titled: Standard and Blocking Goods. A great deal of Chapter 3 is devoted to the multiplicity of w-r relations in decomposable linear production techniques. However most of the analysis is accomplished part by means of numerical examples of 2X2 and 3X3 techniques. The formal characteristics of the multiple normalization commodities that imply linear w-r relations are not presented and only a hint exists that the multiplicity of linear w-r relations disappears if the maximum profit rate of the basic commodities is less than the physical own rate or reproduction of non-basics (p. 74), but a formal proof is absent. An advantage of the analysis of Abraham-Frois and Berrebi in comparison to Vassilakis's article is that they consider cases where $A_{22}$ is decomposable too, and consequently more than two linear w-r relations may appear. Unfortunately, this case is analysed through numerical examples as well.

Finally, Abraham-Frois and Berrebi in Chapter 3 of the book consider the rôle of decomposable techniques in the formation of multiple linear c–g relations. This sort of approach does not exist in Vassilakis's article, but c–g relation is the dual of the w–r and every proposition valid for w–r is valid, under possibly different assumptions, for the c–g. Hence such an analysis of a dual relation could be omitted from a paper that serve no heuristic purposes.

It has to be added that Abraham-Frois and Berrebi write that «... in Sraffa's bean example, the "standard" includes the blocking good (the non-basic good – G.S.), namely beans, and the good used to produce the blocking good, namely the basic good. Thus, in this case, Sraffa's suggestion consisting of only considering the basic sector leads to a mistake» (1997, p. 135). However, Abraham-Frois and Berrebi add that standard commodity's «composition is determined from the peculiar category of blocking goods that we have just defined as those determining the maximum rate of profit (and growth) of the system» (1997, p. 135). The reasons of the so-called peculiarity of the blocking non-basic are not discussed neither in Chapter 3 nor in Chapter 5 of their book.

The only remaining defect so far in the article of Vassilakis is the fact that he assumes $A_{22}$ to be indecomposable. We have to point out that the purpose of his paper is the proof of the existence of standard commodities that include non basic commodities in decomposable techniques, an existence which was considered as not valid in '80s by all authors on linear production systems. If one has accepted the reasoning of Vassilakis and has verified his proofs as true, can easily generalize his results in the case when $A_{22}$ is decomposable.

It must be noted that both papers lack in the analysis of the case when $\lambda_{max}^{A_{11}} = \lambda_{max}^{A_{22}}$ . A thorough analysis can be found in Vouyiouklakis/Mariolis (1992), pp. 143-48, (1993).

Unfortunately the paper of Vassilakis remains unknown and not studied with few exceptions (Stamatis (1984), (1988), Vouyiouklakis/Mariolis (1992)). However, we surmise that Vassilakis's paper still deserves a place in the Sraffian literature and the theory of linear production techniques.

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