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Negative Labour Values and Inferior Processes. A comment

by
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1. Introduction

In the article "Negative Surplus Value and Inferior Processes", Prof. Hosoda asserts that "positive profits coexist with negative surplus value even when the non-inferiority condition is satisfied" [1]. This peculiar result seems to be a direct denial of the well known results of Filippini and Filippini[2] and Fujimori[3]. Also, Hosoda verifies that the Wolfstetter's assertion, that non-inferiority is equivalent to positivity of labour values, is valid only for square 2x2 production techniques.

According to Hosoda all these results are proved by means of a counter-example. In his counterexample Hosoda asserts "each process has an advantage relative to other process. Furthermore, any convex combination of two processes can not produce more net output than another; that is, our technology satisfies non-inferiority. Notice that the technology also satisfies feasibility" [4].

The main argument as stated by Hosoda is that although inferiority does imply non-positivity of labour values, as it is well established, the opposite assertion on the other hand, is not valid, ie. the non-positivity of the labour values does not imply the inferiority of the technology.

2. Definitions of inferiority

At this point we repeat the various definitions of the inferior processes that Hosoda uses in his article.

1. Hosoda (1993), p.33.
2. Filippini/Filippini (1982).
3. Fujimori (1982).
4. Hosoda (1993), p. 37.

Definition 1 (I -Inferiority)

If there exist positive scalars α_i $(i \neq j)$ such that $\sum_{i \neq j} \alpha_i = 1$ and $\sum_{i \neq j} \alpha_i (\mathbf{b}_i - \mathbf{a}_i) > \mathbf{b}_j - \mathbf{a}_j$ then $\mathbf{b}_j - \mathbf{a}_j$ is called I-*inferior* process and $\mathbf{b}_i - \mathbf{a}_i$, $\forall i \neq j$ is considered to be I-*superior* processes to $\mathbf{b}_j - \mathbf{a}_j$.

Definition 2 (F-Inferiority)

Suppose that I and J are subsets of {1, 2, ..., n} such that $I \cap J = \varnothing$. If there exists α_i and β_j, such that

$$\sum \alpha_i \begin{pmatrix} \mathbf{b}_i - \mathbf{a}_i \\ -\boldsymbol{\ell}_j \end{pmatrix} > \sum \beta_j \begin{pmatrix} \mathbf{b}_j - \mathbf{a}_j \\ -\boldsymbol{\ell}_j \end{pmatrix}$$

then processes $\mathbf{b}_j - \mathbf{a}_j$, $j \neq J$ are called F-*inferior*, where $\alpha_i \geq 0$, $\beta_j \geq 0$. If the sign of the inequality is opposite then processes $\mathbf{b}_j - \mathbf{a}_j$, $j \neq J$ are called F-*superior*.

Definition 3 (H-Inferiority)

Suppose that I and J are subsets of {1, 2, ..., n} such that $I \cap J = \varnothing$. If there exists α_i and β_j, such that

$$\sum_{i \in I} \alpha_i \left(\mathbf{b}_i - \mathbf{a}_i \right) > \sum_{j \in J} \beta_j \left(\mathbf{b}_j - \mathbf{a}_j \right)$$

then processes $\mathbf{b}_j - \mathbf{a}_j$ which belong to J are called H-*inferior* processes, where $\alpha_i \geq 0$, $\beta_j \geq 0$, $\sum_{i \in I} \alpha_i = 1$ and $\sum_{j \in J} \beta_j = 1$. If the sign of the inequality is opposite then processes $\mathbf{b}_j - \mathbf{a}_j$ are called H-*superior*.

The definitions of I- and H-*inferiority* are introduced by Hosoda. The definition of F-*inferiority* is introduced by Fujimori (1982).

According to Hosoda the various types of inferiority are not compatible, which is obvious, but his definition of I-inferiority is much broader than H- or F-inferiority and that "... every process can be H-inferior! In other words, every process may, *if wrongly combined*, produce less output than another process" [5].

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5. Hosoda (1993), p. 35.

Obviously this assertion is totally incorrect if we consider as counterexample a special case of joint production, the single production. Every set of single production processes in any combination can not dominate, and can not be dominated by another set of processes that does not belong to the first set. Additionally, Hosoda does not prove the "inefficiency" of H- or F-inferiority criterion, but gives us only an example in order to justify his argument[6]. The example is constructed so that there is no inferior process according definition of I-inferiority although there are inferior processes according definitions of H- and F- inferiority. Due to the fact that this example is quite similar to his central counterexample, we proceed to the kernel of Hosoda's article.

3. Hosoda's main argument

As stated by Hosoda a single process of joint production can and should be compared with any convex combination of the remaining processes in order to prove the inferiority of this single process to the convex combination of the remaining processes. If by any means a convex combination of processes is inferior to another convex combination of processes does not matter, i.e. is totally indifferent to Hosoda. One has to compare only single processes of joint production to any convex combination of the remaining processes for *inferiority only*. If by any chance a single process of joint production is superior to some or to all convex combinations of the remaining joint production processes one has to ignore this fact and proceed comparing the next single process of joint production to any convex combination of the remaining processes for inferiority, ignoring again the fact that it can be superior to some or to all convex combinations of the remaining processes.

Aim of this comment is not to justify or judge the presupposed ability of the individual capitalist nor of a coalition of capitalists nor of a central planner to compare, to classify and to index processes of joint production, ability which is considered to exist according to Hosoda, but presupposing this ability, is to ask why this ability is an 'one way' ability. How can capitalists, or coalitions of capitalists or central planners, see inferiority and can not see the superiority of a single joint production process in comparison to a convex combination of the remaining processes? Only Hosoda can reply to this obvious question.

Proceeding with his counterexample we can see that the net product matrix is

6. Hosoda (1993), p. 36.

$$\mathbf{B}-\mathbf{A}=\begin{bmatrix}12 & 3 & 0\\ 2 & 12 & 0\\ 11 & 11/3 & 3/2\end{bmatrix}-\begin{bmatrix}10 & 0 & 10\\ 0 & 10 & 0\\ 10 & 10/3 & 0\end{bmatrix}=\begin{bmatrix}2 & 3 & -10\\ 2 & 2 & 0\\ 1 & 1/3 & 3/2\end{bmatrix},$$

where **B** is the output matrix **A** is the input matrix. Direct labour inputs are $\boldsymbol{\ell}=(1,1,1)$.

Obviously there is no convex combination of two processes which dominates the remaining third. However, there is a convex combination of two processes that is inferior to, i.e. it is dominated by the remaining third. If we compare an arbitrary convex combination of second and third process to the first we see that:

$$\left\{\begin{array}{c}3\alpha+[-10(1-\alpha)]\leq 2\\ 2\alpha+0(1-\alpha)\leq 2\\ \frac{1}{3}\alpha+\frac{3}{2}(1-\alpha)\leq 1\end{array}\right\}\Leftrightarrow$$

$$\left\{\begin{array}{c}12+13\alpha\leq 0\\ 2\alpha-2\leq 0\\ \frac{7}{6}\alpha-\frac{1}{2}\leq 0\end{array}\right\}\Leftrightarrow$$

$$\left\{\begin{array}{c}\alpha\leq\frac{12}{13}\\ \alpha\leq 1\\ \alpha\geq\frac{3}{7}\end{array}\right\},$$

where $0\leq\alpha\leq 1$.

Consequently, for every $\alpha \in \left(\frac{3}{7}, \frac{12}{13}\right) \subset [0, 1]$ process I dominates processes II and III, that is it is superior to a certain set of convex combinations of processes II and III. Alternatively processes II and III are F- and H-inferior to process I.

Although we have shown that Hosoda's counterexample is deficient only by pointing out that, according to him, capitalists can establish inferiority of a single process of joint production but they are unable to establish superiority of a single process of joint production, we have not determined the error in his argument.

4. Hosoda's misinterpretation of inferiority

Hosoda defines I-inferiority of a single process of joint production in comparison to a convex combination of the remaining joint production processes of the technology. By an analogous definition we define I-superiority of a single process to a convex combination of processes:

Definition 4 (I-superiority)

If there exist positive scalars α_i such that $\sum_{i \neq j} \beta_i = 1$ and $\sum_{i \neq j} \beta_i (\mathbf{b}_i - \mathbf{a}_i) < \mathbf{b}_j - \mathbf{a}_j \quad i \neq j$ then $\mathbf{b}_j - \mathbf{a}_j$ is called a *superior process* and $\mathbf{b}_i - \mathbf{a}_i$ are considered to be inferior to $\mathbf{b}_j - \mathbf{a}_j$.

We can prove easily the following assertion:

Proposition 1

I-inferiority in a technology does not imply I-superiority and inversely.

Proof. In order to prove the inverse part we have to recall the original example of Hosoda. In order to prove that I-inferiority does not imply I-superiority we use the following example:

$$\mathbf{B} - \mathbf{A} = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1/3 & 1 \end{bmatrix}$$

which has an I-inferior process (process III) but not an I-superior because each of the process I and II has an advantage in one commodity and any convex combination of every pair of processes can not be inferior to the third process which establish the non-superiority of I-type. This contributes to the creation of a (grey zone(where I-inferiority and I-superiority do not coexist, where I-inferior processes exist in a technique, while I-superior do not exist. Also, in this 'grey zone' I-superior processes can exist, while I-inferior do not exist. In other words I-superiority is not equivalent to I-inferiority, and I-inferiority is not equivalent to I-superiority as H- or F-inferiority is equivalent to H- or F-superiority. So, with the criterion of H- or F-inferiority there is no 'grey zone' of technologies that use inferior processes but not superior or technologies that use superior process but not inferior, as it exists by using the definitions of I-inferiority and/or I-superiority.

Also one can prove the following assertion.

Proposition 2

I-inferiority implies H-inferiority but the opposite is not true.

Proof. The straight part is trivial. The inverse part can be verified by using the very same example of Hosoda where there is no I-inferiority but as we have seen there is H-inferiority (and F-inferiority). Paradoxicaly, Hosoda concludes from a similar numerical example that his criterion (I-inferiority) "implies inefficiency of a process on the other hand, the latter definition (H-inferiority) refers to the *inefficient combination* of processes which seems quite different from the normal usage of inferiority of a process" and adds "... it must be emphasized that a combination of processes actually adopted in an equilibrium may not be inefficient, even if the technology satisfies H-inferiority... Consequently there is no a *priori* reason for getting rid of H-inferior processes in terms of efficiency" [7]. This assertion of Hosoda is also incorrectly stated. As we shall see in the following, a combination of processes can be inferior at a given profit rate and it can not be inferior at a different value of the profit rate.

Returning to Hosoda's claim that I-inferiority in equilibrium can coexist with positive prices, a positive profit rate and positive activity levels we recall that the authors refered by Hosoda, Filippini and Filippini, define inferiority at a *given profit rate*.

7. Hosoda (1983), p. 36.

We recall now the definition of inferiority and/or superiority given by Filippini and Filippini.

Definition 4 (FF-Inferiority)

Suppose that I and J are subsets of {1, 2, ..., n} such that $I \cap J = \varnothing$. If there exists α_i $i \in I$ and β_j $j \in J$ such that

$$\sum_{i\in I} \alpha_i \begin{pmatrix} \mathbf{b}_i & -(1+r)\mathbf{a}_i \\ & -\boldsymbol{\ell}_i \end{pmatrix} > \sum_{j\in J} \beta_j \begin{pmatrix} \mathbf{b}_j & -(1+r)\mathbf{a}_j \\ & -\boldsymbol{\ell}_j \end{pmatrix}$$

then the processes $j \in J$ are called FF-inferior at *profit rate* r and the processes $i \in I$ are called FF-superior at *profit rate* r.

The difference between the definition of F-inferiority and FF-inferiority is crucial. Inferiority appears and disappears when the profit rate increases from zero to its maximum value. Filippini and Filippini have proved that for a profit rate of 20% there is no FF-inferiority (or FF-superiority) in the original Steedman's counterexample (However, the F-inferiority remains while r increases from zero to its maximum value because r does not affect the inequality). We are going to examine now Hosoda's example for FF-inferiority at profit rate 10%.

$$\mathbf{B}-\mathbf{A}(1+r) = \mathbf{B}-1.1\mathbf{A} = \begin{bmatrix} 1 & 3 & -11 \\ 2 & 1 & 0 \\ 0 & 0 & 3/2 \end{bmatrix}$$

We can easily prove that no convex combination of two processes is FF-superior or FF-inferior to the remaining third process. The non existence of FF-inferiority (or FF-superiority) at profit rate 10% implies by the Filippini/ Filippini[8] theorem that production prices and activity levels are positive at profit rate equal to 10%, a fact that was proved by Hosoda through the direct calculation of production prices and activity levels at profit rate 10%.

Obviously F-inferiority is a misinterpretation by Hosoda of the definition given by Filippini and Filippini and can only confuse someone, because of the fact that a set of processes can be inferior at a given profit rate and can not be inferior at a different value of profit rate. Filippini and Filippini[9] in the french

8. Filippini and Filippini (1982), p. 387.
9. Filippini and Filippini (1984), pp. 60-62.

version of their article have examined the original Steedman's counterexample for FF-inferiority and negative production prices at all values of profit rate from $r = 0$ to $r = r_{max}$ and have proved that inferiority varies within the segment $[0, r_{max}]$.

5. I-inferiority presupposes irrational behaviour rules

Finally, we return to the main argument as stated by Hosoda in order to prove the irrelevancy of his inferiority criterion. Hosoda assumes that every individual capitalist, or coalitions of capitalists, or central planners can compare a convex combination of processes to a single process for inferiority of the single process in comparison to the convex combination of the remaining processes. However, they are unable to compare or verify this single processes for superiority with a convex combination of the remaining processes. Also according to him, capitalists, coalitions of capitalists and central planners, the so-called economic agents, are not capable of comparing a convex combination of process to another convex combination of the remaining process, although they are capable, according to Hosoda, of comparing a convex combination of processes to a *single* process but this ability is limited to the inferiority of this single process only. This incorrectly defined criterion of I-inferiority (and consequently of I-superiority) does not limit the ability of capitalists, coalitions of capitalists and central planners in a realistic manner but it limits their ability in an unrealistic and, I do not hesitate to say, in an irrational one. Their ability to compare processes based on I -inferiority is an 'one way' ability, due to the fact that the so-called economic agents can not verify I-superiority of a single process to the convex combinations of the remaining process, although they can assert the I-inferiority. In mathematical terms there is no such thing as a dual of the binary relation or partial preordering created by Hosoda's definition of I-inferiority, which is not a realistic supposition but, I do not hesitate to repeat, an irrational one.

However, this unique classification of techniques by Hosoda creates a 'grey zone' of techniques, where techniques have the property of I-inferiority and not of I-superiority and where techniques have the property of I-superiority and not of I-inferiority and by selecting a technique from this 'zone' we can construct a counterexample of a technique which has the property of I-superiority but not of I-inferiority and labour values of these techniques are not positive. Also, we can construct an example where a technique has the property of I-inferiority and not of I-superiority and labour values are also non positive[10].

6. Cottrell's contribution to the issue

In a recent issue of *Metroeconomica* Allin Cottrell (Cottrell, 1996) tries to connect Hosoda's results to the production possibility frontier using the notion of reducibility of the net product matrix (**B–A** in our notation). The notion of reducibility of a technique has been introduced by Farjoun and it is strongly related to the notion of F-inferiority and the Fujimori results on negative labour values. Cottrell assures that it is connected to the Filippini and Filippini theorems, which is an inaccurate assertion. Filippini and Filippini theorems are a generalization of Fujimori result, as we have already seen, because the notion of FF-inferiority or inferiority *at a prevailing rate of profit* r is introduced. Farjoun limits his analysis to the case of labour values or in technical terms at a prevailing rate of profit r = 0. From the definition of the term reducible technique, which follows, we may conclude that Farjoun gives an intuitive description of the Fujimori results translating in marxist terms this technical result. According to Farjoun "a reducible table of production processes is a table which allows us to increase total net output without any addition to the total labour and with no new processes introduced. Simply by increasing the level of some processes at the expense of others. In other words, a table is reducible if some rellocation of labour with $\sum_{i=1}^{n} \alpha_i \ell_i = \boldsymbol{\ell}\boldsymbol{\alpha} = 0$ has the property that the associated total net product $\sum_{i=1}^{n} \alpha_i (\mathbf{b}_i - \mathbf{a}_i) = (\mathbf{B}-\mathbf{A})\boldsymbol{\alpha}$ is a non-zero non negative vector" (Farjoun, 1984, p. 41).

<u>Definition 5 (Reducibility)</u>

A joint production technique (**B**, **A**, $\boldsymbol{\ell}$) is called reducible, if there exists a vector $\boldsymbol{\alpha}$ such that $\boldsymbol{\ell}\boldsymbol{\alpha} = 0$ and $(\mathbf{B}-\mathbf{A})\boldsymbol{\alpha} \geq 0$.

(The notation of Farjoun and Cottrell has been modified, in order to be in accordance with our previous notation). Cottrell, also, defines reducibility as a property of the transpose of **B–A**, but we proceed with the matrix **B–A** and not

10. See our numerical example in page 63.

its transpose[11]. Farjoun and Cottrell surmise that reducibility of a technique implies non-positivity of labour values, and strict positivity of labour values implies non-reducibility but only a hint of the proof can be found in Cottrell (Cottrell, 1996, p. 73). Formally both of them assert:

Proposition 3

A reducible joint production technique implies non-positive labour values and non-positive labour values imply reducibility of a technique. In other words

either

$$\text{(I)}\quad (\mathbf{B}-\mathbf{A})\boldsymbol{\alpha} \geq 0 \text{ and } \boldsymbol{\ell\alpha} = 0 \text{ hold}$$

or

$$\text{(II)}\quad \mathbf{w}(\mathbf{B}-\mathbf{A}) = \boldsymbol{\ell} \text{ and } \mathbf{w} \geq 0 \text{ hold,}$$

where **w** denotes the vector of uniform labour values.

Proof: We prove first that (I) and (II) are not compatible. Premultiplying $(\mathbf{B}-\mathbf{A})\boldsymbol{\alpha}$ by $\mathbf{w}$, $\mathbf{w} \geq 0$, we get $\mathbf{w}(\mathbf{B}-\mathbf{A})\boldsymbol{\alpha} > 0$, but postmultiplying $\mathbf{w}(\mathbf{B}-\mathbf{A})\boldsymbol{\alpha} = \boldsymbol{\ell}$ by $\boldsymbol{\alpha}$ we get $\mathbf{w}(\mathbf{B}-\mathbf{A})\boldsymbol{\alpha} = \boldsymbol{\ell\alpha} = 0$, which is inconsistent with $\mathbf{w}(\mathbf{B}-\mathbf{A})\boldsymbol{\alpha} > 0$. In order to complete the proof we apply Farkas lemma.

Farkas lemma

For every fixed mXn matrix $(\mathbf{B}-\mathbf{A})$ and for every fixed 1Xn row vector $\boldsymbol{\ell}$ either

$$\text{(I}'\text{)}\quad (\mathbf{B}-\mathbf{A})\mathbf{x} \leq 0 \text{ and } \boldsymbol{\ell}\mathbf{x} > 0 \text{ has a solution}$$

or

$$\text{(II}'\text{)}\quad \mathbf{w}(\mathbf{B}-\mathbf{A}) = \boldsymbol{\ell} \text{ and } \mathbf{w} \geqq 0 \text{ has a solution,}$$

but never both.

Suppose that (I′) is valid and let $\boldsymbol{\ell}\mathbf{x} = \xi > 0$. In addition technique $(\mathbf{B}, \mathbf{A}, \boldsymbol{\ell})$ is by definition feasible or productive, namely there exists a positive column vector of activity levels such that the implied or related net product is semipositive. In mathematical terms, $\exists \mathbf{y} > 0$, such that $(\mathbf{B}-\mathbf{A})\mathbf{y} \geq 0$. Also, we assume that labour is indispensable in every production process, i.e. $\boldsymbol{\ell} > 0$.

11. Farjoun does not normalize processes, in the contrary Cottrell adapt implicitly the well known normalization $l_i = 1, i = 1, 2, \ldots, n$. This implicit assumption of Cottrell can lead to misinterpretations because α is not defined in Cottrell's article as the re-allocation coefficents of labour, as it is defined in Farjoun's article, but as arbitrarily defined numbers. The validity of the Farjoun's result, however, remains intact, but its heuristic purpose is lost.

Productivity of the technique and indispensability of labour imply $\mathbf{y} > 0$, such that $\boldsymbol{\ell}\mathbf{y} > 0$. Due to the fact that vector $\mathbf{y}$ is not uniquely determined, because for every positive scalar $\mathbf{y}$ and $\boldsymbol{\ell}\mathbf{y} = \xi > 0$. Consequently we can choose without lose of generality, such that $\boldsymbol{\ell}\mathbf{y} = \xi > 0$. Denote $\boldsymbol{\alpha} = \mathbf{y} - \mathbf{x}$. Obviously the following relations are valid:

$$(\mathbf{B}-\mathbf{A})\boldsymbol{\alpha} = (\mathbf{B}-\mathbf{A})\mathbf{y} - (\mathbf{B}-\mathbf{A})\mathbf{x} \geq 0 \text{ and } \boldsymbol{\ell}\mathbf{a} = \boldsymbol{\ell}\mathbf{y} - \boldsymbol{\ell}\mathbf{x} = 0.$$

Thus, (I′) implies (I). By a similar argument we can prove that (I′) implies (I). Consequently, (I) and (II) are equivalent, and from Farkas lemma we have that (II) is not valid.

However (II′).asserts that $\mathbf{w} \geqq 0$, but from the indispensability of labour, i.e. $\boldsymbol{\ell} > 0$ and from the relation $\mathbf{w}(\mathbf{B}-\mathbf{A}) = \boldsymbol{\ell} > 0$ we can exclude the case $\mathbf{w} = 0$. We have proved that $\mathbf{w} \geq 0$, but not $\mathbf{w} > 0$ as Cottrell and Farjoun assert. In order to get this result additional assumptions on $(\mathbf{B}-\mathbf{A})$ must be introduced[12].

12. For example consider the following production technique net product matrix and labour input vector:

$$\mathbf{B}-\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \text{ and } \boldsymbol{\ell} = (1,1,1).$$

The following vectors $\mathbf{w}_1 = (1,0,0)$ and $\mathbf{w}_2 = \left(0, \frac{1}{2}, \frac{1}{2}\right)$ satisfy the equations that determine the uniform labour values, i.e. $\mathbf{w}(\mathbf{B}-\mathbf{A}) = \boldsymbol{\ell}$ or

$$(\mathrm{w}^1, \mathrm{w}^2, \mathrm{w}^3) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} = (1,1,1).$$

On the other hand equations $(\mathbf{B}-\mathbf{A})\boldsymbol{\alpha} \geq 0$ and $\boldsymbol{\ell}\boldsymbol{\alpha} = 0$ are inconsistent, due to the fact that:

$$(\mathbf{B}-\mathbf{A})\boldsymbol{\alpha} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 \\ 2\alpha_2 + \alpha_3 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha_2 - \alpha_1 \\ \alpha_1 - \alpha_2 \end{bmatrix}.$$

Numbers $(\alpha_1 - \alpha_2)$ and $(\alpha_2 - \alpha_1)$ are of opposite sign, and the trivial case $\alpha_1 = \alpha_2$ can be excluded because implies $(\mathbf{B}-\mathbf{A})\boldsymbol{\alpha} = 0$. Hence $(\mathbf{B}-\mathbf{A})\boldsymbol{\alpha} \geq 0$ and $\boldsymbol{\ell}\boldsymbol{\alpha} = 0$ are inconsistent for the above value of $(\mathbf{B}-\mathbf{A})$. Additionally the technique above is feasible, due to the fact that for every $\mathbf{x} > 0$, $\mathbf{x} \in \mathbf{R}^3$, $(\mathbf{B}-\mathbf{A})\mathbf{x} > 0$, and labour is indispensible, i.e. $\boldsymbol{\ell} > 0$. However $\mathbf{w}$ is not necessarily positive, but it is proved, by means of our proposition 3, to be only semipositive although both Farjoun and Cottrell assert that $\mathbf{w}$ is strictly positive. The reason of the

Finally, we have to add that the required result, $\mathbf{w} \geq 0$, is established by means of the *implicitly* assumed productivity of the technique ($\mathbf{B}$, $\mathbf{A}$, $\boldsymbol{\ell}$) and indispensability of labour $\boldsymbol{\ell} > 0$, necessary conditions for the proof of Proposition 3. Both Cottrell and Farjoun do not mention the above conditions and write as if reducibility implies non positive labour values even in non-productive techniques using automatization, i.e. production processes with no labour inputs.

Consequently, the notion of reducibility can be omitted, at least from our comment, due to the fact that its analysis and interpretation is beyond our scope here, and that we can proceed using only the Filippini and Filippini result.

Cottrell asserts that "... Steedman's negative labour values are an interesting curiosity, arising only in an inefficient economy (i.e. one operating inside its production possibility frontier)" (Cottrell, 1996, p. 71). However, this assertion is not correct. A linear joint production technique operating at a profit rate r can be on the efficient production frontier[13], although the same technique can be inside the efficient production frontier, when it operates the same processes as before at a different profit rate (for example, in Steedman's example, when profit rate equals zero). The figure 1 shows the efficient production frontier of Steedman's counterexample at four different values of the profit rate: $r_0=0$, $r_1=.10$, $r_2=.20$ and $r_3=r_{max}=.444949...$ (figure 1 is adapted from Abraham–Frois/Berrebi, 1997, p. 26). Line segments $M_1^0\,M_2^0$, $M_1^1\,M_2^1$, $M_1^2\,M_2^2$, $M_1^{max}\,M_2^{max}$, correspond to profit rates r_0, r_1, r_2, r_{max} respectively.

As it can be easily shown the efficient production frontier has a positive slope at and it is the line segment joining points M_1^0 (1,1) and M_2^0 (3,2). Obviously the efficient production frontier is degenerated to the point M_2^0(3,2) given the fact that process II has a greater net product than process I. So, when $r = 0$ process I is shut down and economy operates only process II. However, at a prevailing profit rate $r > .10$ the efficient production frontier changes slope, which becames negative. That implies the operation of both processes simultaneously, given the fact that the line segment joining the points M_1^r (1–5r, 1) and

semipositivity of $\mathbf{w}$ in our numerical example, and the reason of the non-uniqueness of the solutions $\mathbf{w}$ of the equations $\mathbf{w}(\mathbf{B}-\mathbf{A}) = \boldsymbol{\ell}$, is the linear dependence of the columns of ($\mathbf{B}$–$\mathbf{A}$) matrix, a case which is not discussed either in Farjoun or in Cottrell.

13. The terms production possibility frontier and efficient production frontier will be used alternatively as if they are equivalent.

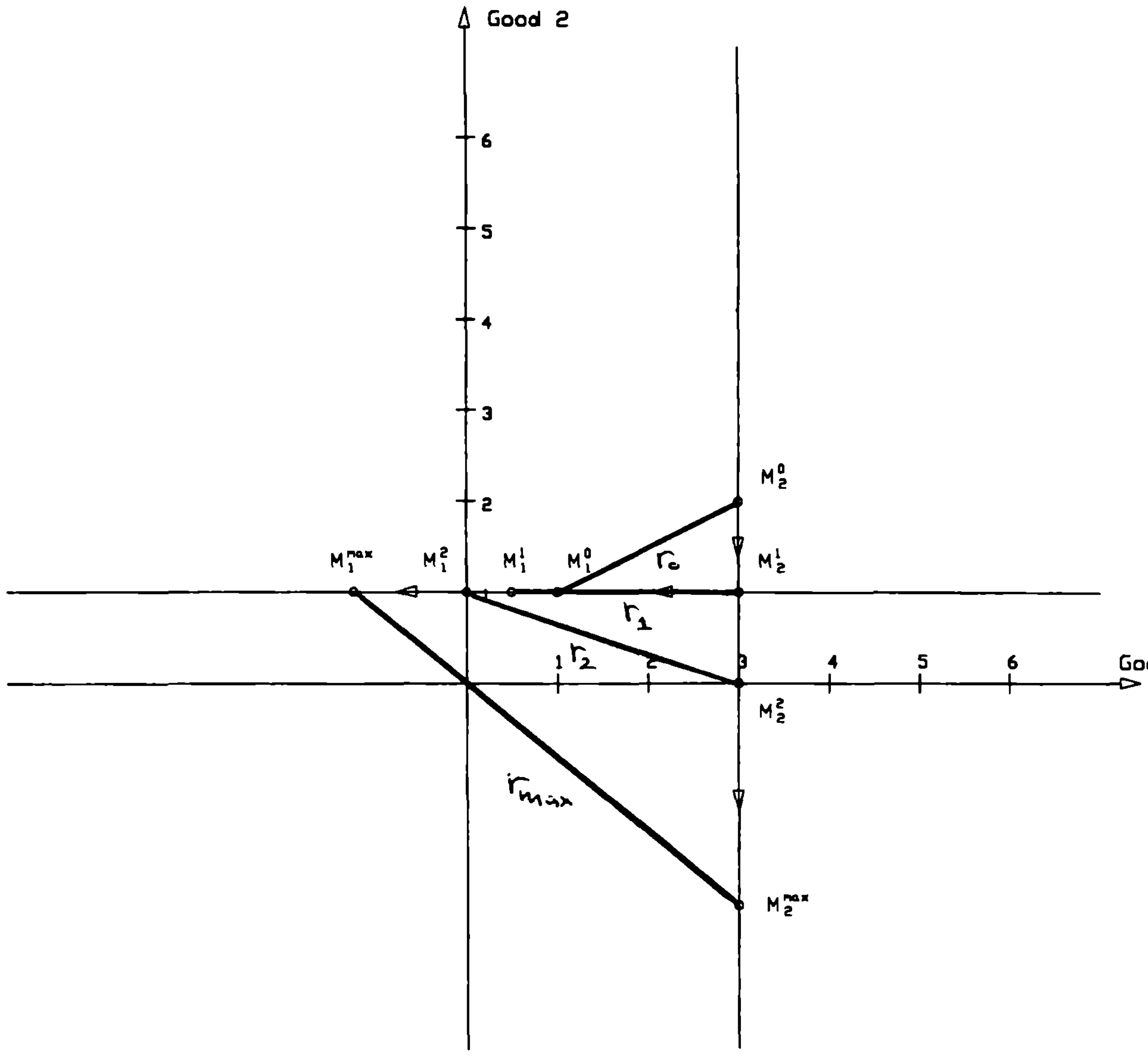

Figure 1

Efficient production frontier in Steedman's numerical example for various values of the prevailing profit rate; $r_0=0$, $r_1=.10$, $r_2=.20$ and $r_{max}=.444949...$

M_2^r (3, 2–10r) has a negative slope for r > .10. In Steedman's counterexample the production possibility frontier retains its negative slope for .10 < r ≤ .444949... = r_{max}.

Hosoda's counterexample has a degenerated to a single point production possibility frontier as well. The point (2, 2, 1) describes the efficient production frontier at r = 0, given the fact that process I has greater net output than a convex combination of processes II and III. So, processes II and III are shut down in Hosoda's counterexample at r = 0, if one operates technique at its efficient production frontier, which is degenerated to a point. At a prevailing profit rate r = .10, instead, the picture changes and all three processes operate, because FF-inferiority is vanished from Hosoda's technique. Figures 2-4 show

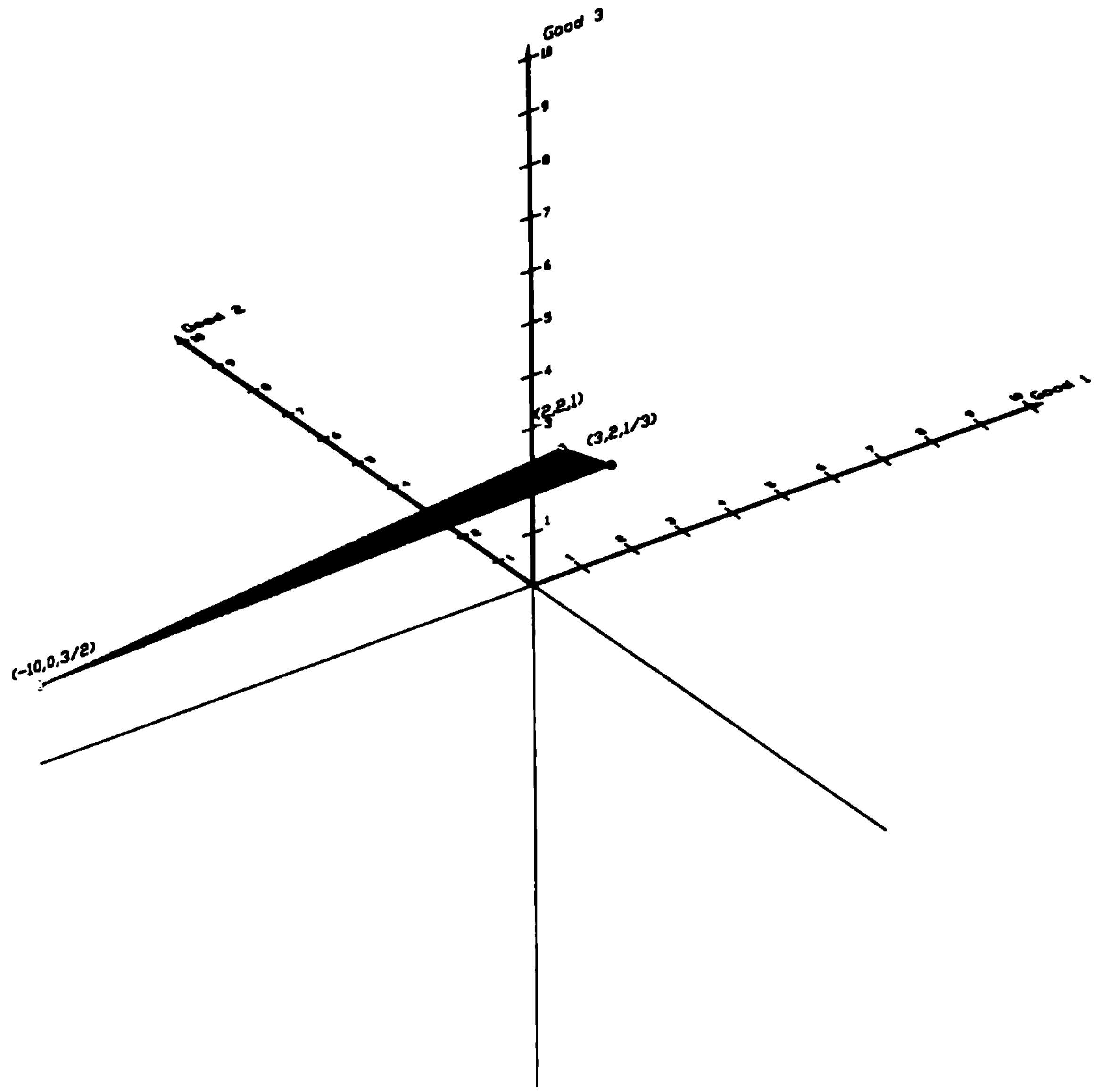

Figure 2

Efficient production frontier in Hosoda's numerical example at a prevailing rate of profit r=0.

the production possibility frontier. When r = 0 the production possibility frontier is the vertex (2, 2, 1) of the polyedron (triangle). When r = .10 the efficient production frontier is the whole triangle. In this case there is no way to compare processes I-III for FF-inferiority because their r-net products are incompatible. Also, processes I and II produce an r-net product of commodities 1 and 2 and process III produces and r-net product of commodity 3 in a positive quantity. Thus processes I, II and III must operate simultaneously at a prevailing profit rate r = .10.

In a further step and trying to resolve Hosoda's paradox Cottrell re-introduces the so-called 'rational' values (or prices) which are introduced back in the 70s by von Weiszäcker and Samuelson. The judgment of the importance

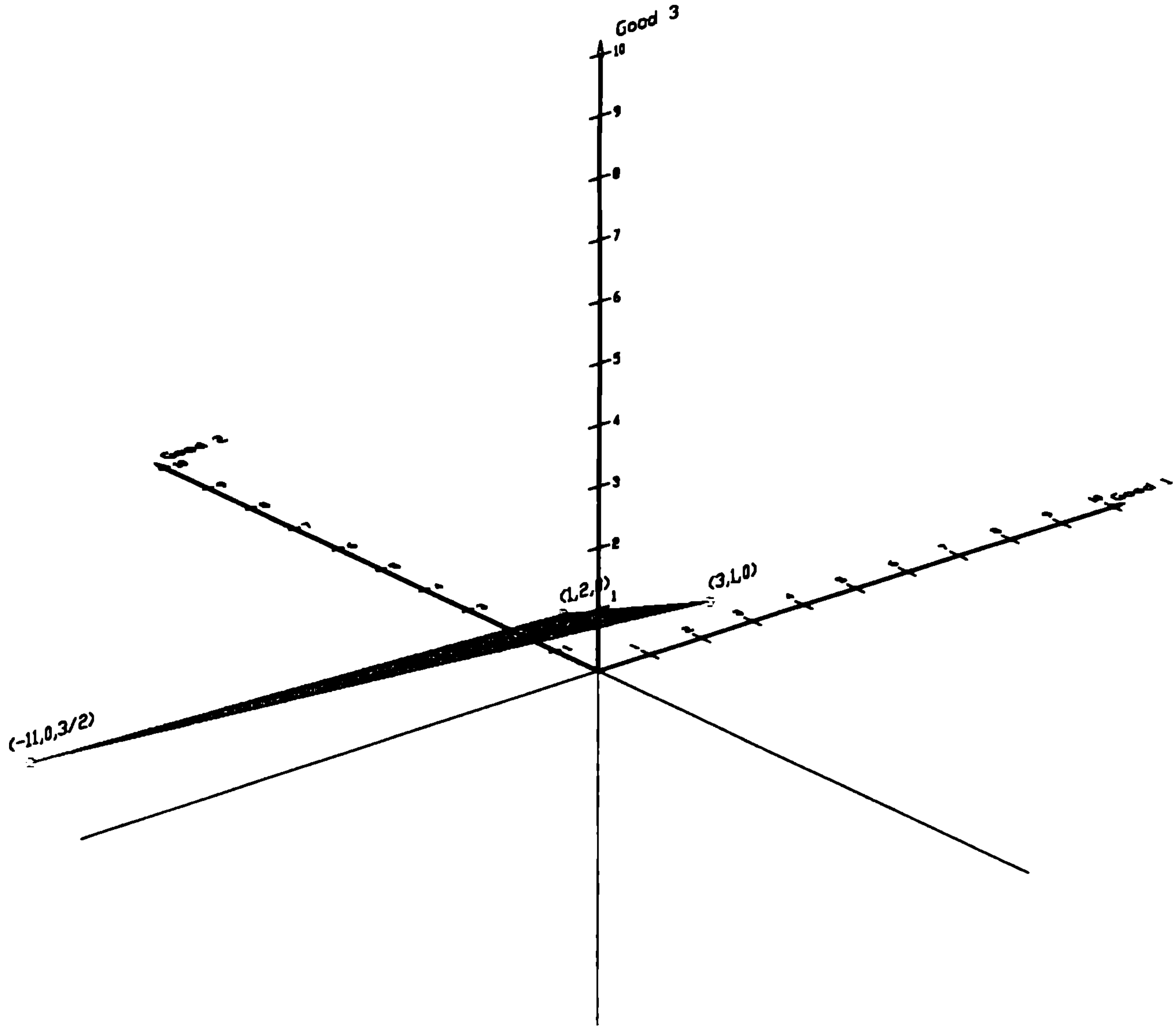

Figure 3
Efficient production frontier in Hosoda's numerical example at a prevailing rate of profit r=.10.

of these magnitudes to the planning of the central planned economies is totally beyond our scope here. However, we must note that these 'rational' values (or prices, as alternatively and wisely von Weiszäcker and Samuelson call them) are production prices at a given profit rate, and specifically, at a profit rate equal to the growth rate, which might be smaller than the actual profit rate. In addition, these 'rational' values can be negative when the rate of growth is small enough and close to zero, as Cottrell points out. These magnitudes might serve as analytical tools but they are not labour values. As Steedman has surmised and Filippini and Filippini have verified inferior processes can be used at a certain profit rate (or growth rate), because they stop to be FF-inferior at this rate. That implies the positivity of production prices or, of the

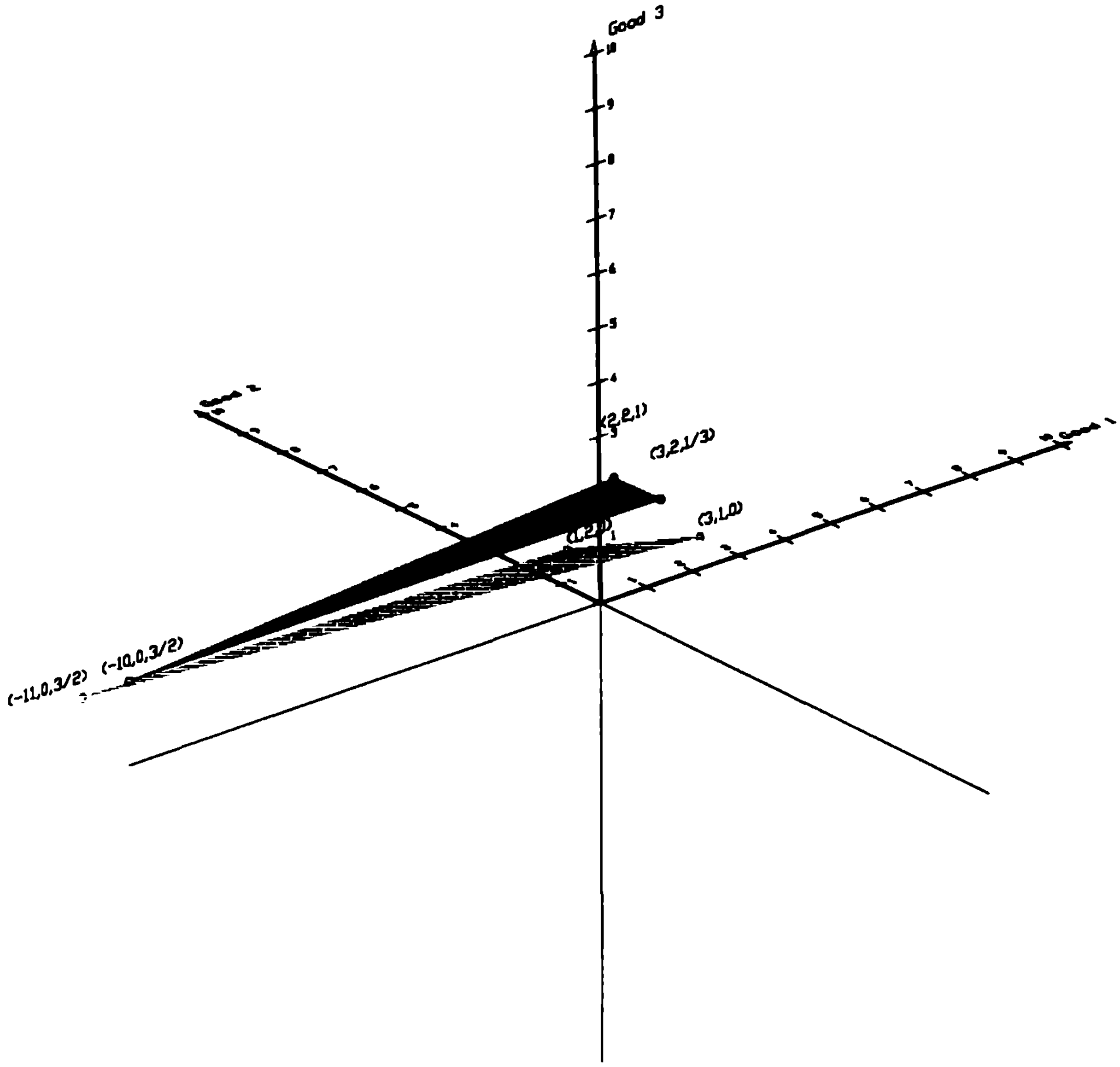

Figure 4

Efficient production frontier in Hosoda's numerical example at prevailing rate of profit r=0 (upper simplex) r=.10 (lower simplex).

'rational' values or prices at this profit rate (or growth rate). This explains the fact why I-, H- or F- inferior processes can be used in 'equilibrium' (at a certain value of profit rate and/or growth rate), which remains a mystery for Hosoda.

Another point of Cottrell's note worths mentioning. According to him "... from the fact that processes II and III in the example above (Hosoda's original example) are jointly F-inferior to process I, it does *not* follow that we can delete II and III from the table. In fact it is clear from the inspection of (**B**–**A**) that in order to maximize output of commodity 2 (respectively 3) process II (respectively III) must be used... In the example one may use process II or process III in conjunction with process I... and of course each of these 2x3 tables (Cottrell means systems of two equations with three unknowns – G.S.)

has a strictly positive solution for the labour values". Cottrell gives us a pair of labour value vectors as numerical solutions, each one of them satisfies one of the pair of systems of two equations with three unknowns. Cottrell omits, however, to let us know that a system of two linear equations and three unknowns is not fully determined and has usually infinite solutions. Under certain conditions the set of solutions of a system of two linear equations with three unknowns intersects the $\mathbf{R}^3_{++} = \{x | x \in \mathbf{R}^3, x > 0$, or in other words has infinite parametrically determined solutions that belong to a not closed subset of $\mathbf{R}^3_{++}$. For example processes I and II have the following labour value equations:

$$2w_1 + 2w_2 + w_3 = 1,$$

$$3w_1 + 2w_2 + \tfrac{1}{3}w_3 = 1,$$

subject to

$$(w_1, w_2, w_3) > 0.$$

The solution set is $\left(\frac{2}{3}\beta, \frac{3-7\beta}{6}, \beta\right) \subset R^3 : \beta \in R$ and the set of positive solutions is $\left(\frac{2}{3}\beta, \frac{3-7\beta}{6}, \beta\right) : \beta \in \left(0, \frac{3}{7}\right)$. Similarly, the set of labour value equations for processes I and II is:

$$2w_1 + 2w_2 + w_3 = 1,$$

$$-10w_1 + \tfrac{3}{2}w_3 = 1,$$

subject to

$$(w_1, w_2, w_3) > 0.$$

The set of solutions is $\frac{1}{20}\left(3\beta-2, 12-13\beta, 20\beta\right) \subset R^3 : \beta \in R$ and the set of positive solutions is $\frac{1}{20}\left(3\beta-2, 12-13\beta, 20\beta\right) : \beta \in \left(\frac{2}{3}, \frac{12}{13}\right)$. Obviously the two subsets of positive solutions of the two systems do not intersect. Cottrell gives us as solution for the equations that consist of process I and II the vector $\left(\frac{1}{9}, \frac{2}{9}, \frac{1}{3}\right)$. This vector satisfies only the equation of process I but not that of process II. Correct solutions are vectors $\left(\frac{2}{9}, \frac{1}{9}, \frac{1}{3}\right)$, $\left(\frac{1}{9}, \frac{11}{36}, \frac{1}{6}\right)$ etc[14].

Although Cottrell has the best intentions for the matter, fails to reveal the error in Hosoda's argument, which is the incorrectly defined notion of I-inferiority. The introduction of the so-called 'rational' values or prices and of the abnormalities of the production possibility frontier in Hosoda's example does not help us to realise the irrational behaviour pattern that Hosoda presupposes in the definition of I-inferiority. This presupposition is the crux of the abnormality of negative labour values with no inferior processes.

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14. It must be noted that there is no strict negative labour value vector that satisfy either equations of processes I and II, i.e. at least one labour value is positive in feasible or productive joint production techniques. See Wolfstetter (1976).

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