POLITICAL ECONOMY - 10 - SPRING 2002 - p.p. 5 - 16

Ἄρθρα - Articles

The Un-Simple Analytics of Temporal Value Calculation*

by
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I

The classical-Marxian tradition proposes a measure of value, as distinct from money, that would be present even in the extreme case of a one-good, aggregative macromodel. This is, of course, the quantity of labor, or labor-time, needed to produce a unit of goods. "All" one needs to make this operational –in addition to the mysterious all-purpose single good– is the assumption that labor is homogeneous with regard to skill, efficiency, and effort. Production can then be represented by a flow of material input M, a flow of labor L cooperating with and processing that input, and a corresponding flow of output X:

$$M, L \rightarrow X$$

The *unit value* of the (gross) output is the sum of direct and indirect labor used to produce one unit, where the indirect labor is the labor embodied in the material input M. Since this last amount of labor is also a sum of direct and indirect components, we have an infinite regress, which however corresponds to a very simple simultaneous calculation. With λ representing the (unknown) unit value, the equation

$$\lambda M + L = \lambda X$$

* I wish to acknowledge the expert mathematical assistance of Professor Joseph Krieger, of the Physics Department, Brooklyn College, City University of New York, who contributed significantly to the solution of the problem and formulation of the results with which this paper is concerned. Useful comments from referees are also acknowledged. The author is, as always, alone responsible for the finished product.

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has the solution

$$\lambda = \frac{L}{X-M} = \frac{L/X}{1-M/X} = \frac{1}{1-\alpha},$$

where the first equality shows λ to be current labor divided by the *net* product, $X-M$, and the second and third replace the absolute quantities with the more analytically useful technological ratios: unit labor input, l, and unit material input, α.

This formalization has a venerable lineage in economic thought (see, *e.g.*, Dobb, 1955; Meek, 1956. Brody, 1970), as it represents a point of intersection between the "micro" and "macro" worlds of economics. To be sure, one can study the *relative price* aspect of valuation using real relative prices or money prices in a model with multiple goods, without reference to an aggregative framework. One can also examine macroeconomic processes without invoking the value concept: as long as we are willing to use the aggregate ("one good") model at all, why assume the additional burden of λ, when quantities of real output-cum-income, X, and (if desired) quantities of money in its own dimension are all that we need to describe aggregate "real" and price phenomena? Finally, to complete the list, I note the existence of a Marx-derived school of thought that sees value –the qualitative embodiment of abstract labor– as a key to unraveling the mysteries of market-shrouded social relations. Again, and whatever one makes of this line of thought, *calculation* of a macroeconomic labor value coefficient, λ, is clearly irrelevant to it.[1]

Nevertheless, the idea of a quantity of labor embodied in goods carries a strong intuitive appeal that has attracted economists in every era, especially those working in the classical tradition. The value concept provides a manageable way to address certain kinds of micro phenomena, such as the tension between perceived and realized rates of return, in an aggregative framework, without the entire apparatus of relative prices (see Laibman, 1997). It makes sense to complete the theory toolkit with the most rigorous possible version of the static λ-determination model.

The classical tradition, however, is practically *defined* by dynamic assumptions: in particular, technical change and growth as the unavoidable

1. My own efforts to address this question in general will be found in Laibman, 1992, chapters 1-3, and, most recently, in a forthcoming paper on the problem of labor value in a multi-commodity, albeit static, context; Laibman, 2002.

starting point for every inquiry into economic phenomena, including those concerned with value formation. But when we try to define and solve a dynamic equation to determine λ in the presence of continuous technical change, we find that the matter is much more complex than the simple formalization above implies. This paper is devoted to analysis of this problem, which has not, to this writer's knowledge, been posed, let alone solved, previously.

II

The dynamic assumption examined in this paper is quite simple: technical change is taking place. In the one-good world, this is represented by continual change in the two technical coefficients, α and l. This means, of course, that we have to rethink the static formalization of section I.

Begin by writing

$$M(t), L(t) \Rightarrow X(t+1). \tag{1}$$

This is a dynamic (in the limited sense required here) version of the earlier representation of production, but with time subscripts added. Inputs occur at time *t*, while output appears at *t + 1*. We thus have a period (discrete time) formulation, in which production takes time: inputs and outputs must be carefully time dated, with inputs valued and applied in one period, and output resulting in the subsequent one.

The technical coefficients, reformulated to correspond to this dynamic picture, are:

$$\alpha_t = \frac{M(t)}{X(t+1)} \quad l_t = \frac{L(t)}{X(t+1)}.$$

The period 0 values are $\alpha_0 = M(0)/X(1)$, and $l_0 = L(0)/X(1)$; note that X(0) does not play an explicit role. The fundamental value equation posits dynamic production, with unit value also changing through time:

$$\lambda(t+1)X(t+1) = \lambda(t)M(t) + L(t)$$

$$\lambda(t+1) = \lambda(t)\frac{M(t)}{X(t+1)} + \frac{L(t)}{X(t+1)}$$

$$\lambda(t+1) = \lambda(t)\alpha_t + l_t \tag{2}$$

Eq. (2) describes value formation as a process operating in time, in which input values and output values differ as a result of technical change. The fact that it is production, as opposed to realization, that inherently takes time is embodied in the lag structure: inputs at one period, outputs in the next, with realization –and therefore unit value formation– occurring at the moment at which output appears.

By induction over *t* in (2), and without imposing at this point any particular assumptions about the time paths of the technical coefficients, we obtain:

$$\lambda(t) = \lambda(0)\alpha_{t-1}\ldots\alpha_0 + \sum_{\tau=1}^{t} \alpha_{t-1}\ldots\alpha_\tau l_{\tau-1} \qquad (3)$$

The reader can verify that this form generates the entire set of $\lambda(t)$ values such as $\lambda(1) = \lambda(0)\alpha_0 + l_0$ and $\lambda(2) = \lambda(0)\alpha_1\alpha_0 + \alpha_1 l_0 + l_1$, given the convention that the product of zero terms $\equiv 1$ ($\alpha_{i-1}\alpha_i \equiv 1$).

We will examine different assumptions about the time path of α_t, including the postulate of rising α, or "capital-deepening" technical change (see below). I begin, however, with the case of progressive technical change, in which both input coefficients are falling. (Falling l_t, or rising labor productivity, is assumed throughout.) The most general progressive assumption is that the two input coefficients, α_t and l_t, fall ultimately: $\alpha_t, l_t \to 0$ as $t \to \infty$. This is enough to guarantee that the first term in (3) eventually approaches zero. For the second term we need the slightly stronger assumption on the material input coefficient that $\alpha_{t-1} \leq \alpha_t < 1$, for all t. This term can be written:

$$\alpha_{t-1}\sum_{\tau=1}^{t-1} \alpha_{t-2}\ldots\alpha_\tau l_{\tau-1} + l_{t-1},$$

The summation in the first term of this expression cannot be greater than

$$\max(l_0, \ldots, l_{t-2})(\alpha_1^{t-2} + \ldots + \alpha_1 + 1) =$$

$$= l_{\max}\frac{1-\alpha_1^{t-1}}{1-\alpha_1}.$$

The second term in (3) therefore cannot be greater than

$$\alpha_{t-1}\frac{l_{\max}}{1-\alpha_1} + l_{t-1},$$

which clearly vanishes as $t \to \infty$. So we know that the temporally correct unit value coefficient approaches zero under continuous progressive technical change – something that in itself is no surprise.

Further analysis of the time path of λ, however, will contain some surprises; to find these requires a somewhat stronger restriction on the path of technical change. In what follows, I will adopt the simplest possible assumption of constant proportional change in the technical coefficients; this assumption, after all, is the systematic expression of the case in which the rate of change is constant *on average*, over long periods of time. We then define *growth factors* of α and l:

$$G_\alpha = \frac{\alpha_{t+1}}{\alpha_t}, \quad G_l = \frac{l_{t+1}}{l_t},$$

resulting in the dynamic equations

$$\alpha_t = \alpha_0 G_\alpha^t, \quad l_t = l_0 G_l^t$$

from which, finally, we have a fully explicit form of the difference equation for the time path of unit value:

$$\lambda(t+1) = \lambda(t)\alpha_0 G_\alpha^t + l_0 G_l^t. \tag{4}$$

It should be noticed that growth *factors* (unlike growth *rates*) are always non-negative; are pure numbers; and are equal to unity when no growth (change) is taking place. Technical progress, as before, lowers the labor input per unit of output over time: $G_l < 1$. $G_\alpha > 1$ represents the case of a rising "composition of capital," or capital deepening[2]; $G_\alpha = 1$ is the case of neutral

2. The absence of capital *stocks* in this model will be noticeable. Particularly with a view to studying industrial production, the choice of a pure circulating-capital model would seem to be unwarranted. I make this assumption only because it is considered to be the most general form of classical models of production (see various entries in Kurz and Salvadori, 1998). In other work, I have used its opposite, the pure fixed capital case, as well as hybrid models containing both fixed and circulating capital.
3. In a classical production model of this type, neutrality and bias of technical change are measured only with respect to the trend of α_t. There are no continua of techniques at each moment of time, and no marginal products; the distinctions among various definitions of neutrality that emerge in neoclassical contexts –Harrod, Hicks, Solow, Uzawa, etc.– therefore do not arise here (see, *i.a.*, Wan, 1971, ch. 5).

and variable (time-dependent) non-homogeneous term. That continuous-time equation has a general solution, as is well known. However, the integrals involved in the continuous counterpart to (4) are non-algebraic, which suggests that no closed-form representation of the solution is possible. I therefore pursue the matter in the form of the discrete-time difference equation (4). This formulation also has the advantage of facilitating confirmation of results by numerical simulation. I note at the outset that a mathematically "interesting" problem emerges from nothing more than postulating technical coefficients that change at a constant rate over time.

III

We may pursue the solution to (4) as follows.

Begin by taking the simplest possible case and eliminating technical change altogether: $G_\alpha = 1$ and $G_l = 1$. (4) then reduces to $\lambda(t+1) = \lambda(t)\alpha + 1$, a simple first-order difference equation, whose solution is

$$\lambda(t) = \left[\lambda(0) - \frac{1}{1-\alpha}\right]\alpha^t + \frac{1}{1-\alpha}. \tag{5}$$

The second term corresponds to the "particular integral" of ordinary differential equations; $\frac{1}{1-\alpha}$ is the value for λ predicted by the static model. The adjustment of the arbitrary initial unit value, $\lambda(0)$ toward its center, or equilibrium, is evident (we may think of $\lambda(0)$ as some sort of "marker" unit value, although the distinction between market and equilibrium values is admittedly somewhat abstract in a one-good model). The first term –the "complementary function"– vanishes over time, since $\alpha < 1$ in a productive economy, and $\lambda(t)$ converges to a horizontal asymptote.

We next allow technical progress to occur, although at first we restrict it to a neutral form: $G_\alpha = 1$, $G_l < 1$. Eq. (4) now takes the form $\lambda(t+1) = \lambda(t)\alpha_0 + l_0 G_l^t$. The solution, whose derivation follows by analogy with the previous case (we will obtain it formally below) is:

$$\lambda(t) = \left[\lambda(0) - \frac{l_0}{G_l - \alpha_0}\right]\alpha_0^t + \frac{l_0 G_l^t}{G_l - \alpha_0}. \tag{6}$$

technical change[3]; and $G_\alpha < 1$ is capital-"shallowing."

The first term of (6) again is the "complementary function"; it exists due to $\lambda(0) \neq \frac{l_0}{G_l - \alpha_0}$ the deviation of unit value at time 0 from its central tendency at t = 0. As before, this term vanishes as $t \to \infty$. The second term is the "particular integral": the asymptotic target toward which λ converges, a target that is now itself falling due to technical progress. The first term vanishes much more rapidly, since it is reasonable to assume $\alpha < G_l$, justifying the different interpretations given to the two terms.

Eq. (6) establishes, for the case of neutral technical change, a general principle of motion for λ. The unit value must eventually fall, when unit labor input is falling at some constant rate. It will rise, beginning at t = 0, when $\lambda(0) < \frac{l_0}{G_l - \alpha_0}$; with the rapid disappearance of the first term, however, its movement must eventually be dominated by the second term, also falling. Nothing in this most temporally rigorous account falsifies the intuition from the static formulation with which we began: falling l (with constant α) means falling λ. The dynamic version, however, does allow us to examine "out-of-equilibrium" movement,[4] when λ(0) is not equal to its benchmark level, $\frac{l_0}{G_l - \alpha_0}$.

Moreover, that benchmark level is not the same as the one emerging from the static model: $\frac{l_0}{G_l - \alpha_0} > \frac{l_0}{1 - \alpha_0}$. This is a genuine temporal result: while the trend of λ can be predicted from the static formulation, its level is permanently higher than the level predicted from that same formulation.

4. It should perhaps be reemphasized that the macro unit value concept, λ, makes possible the paradoxical: analysis of value without exchange-value. The existence of a value coefficient as such implies the possibility of discrepancy between actual and benchmark quantities, as also between momentary (innovators') and ultimate (realized) returns (see section I), without explicit representation of exchange-value and price in a multi-good framework. The λ(*t*) story is a concentrated expression of a fully adequate multi-commodity model; since the properties of this model are still not fully understood even in a static context (see Laibman, 2002), study of its aggregative counterpart would appear to be justified.

We now turn to the most general form of the dynamic value equation, eq. (4): both technical coefficients change over time. With a constant growth factor of α, $G_\alpha \neq 1$, we have $\alpha_k = \alpha_0 G_\alpha^k$, and the product in the first term of (3) can be written as

$$\alpha_{t-1} \ldots \alpha_0 = \alpha_0 G_\alpha^{t-1} \ldots \alpha_0 G_\alpha^0$$

$$= \alpha_0^t G_\alpha^{t-1+t-2+\ldots+1+0}$$

$$= \alpha_0^t G_\alpha^{(1/2)t(t-1)}.$$

For the second term of (3), we evaluate

$$\alpha_{t-1} \ldots \alpha_\tau = \alpha_0 G_\alpha^{t-1} \ldots \alpha_0 G_\alpha^\tau$$

$$= \alpha_0^{t-\tau} G_\alpha^A$$

where $A = [(t-1) + (t-2) + \ldots + \tau]$.

The sum A can be further processed

$$A = \tau + (\tau+1) + \ldots + (t-1)$$

$$= \tau + (\tau+1) + \ldots + (\tau + t - \tau - 1)$$

$$= \underbrace{[\tau + \ldots + \tau]}_{t-\tau \text{ terms}} + \underbrace{[1 + 2 + \ldots + (t-\tau-1)]}_{t-\tau-1 \text{ terms}}$$

$$= \tau(t-\tau) + \frac{1}{2}(t-\tau-1)(t-\tau)$$

$$= \frac{1}{2}[(t^2-t)-(\tau^2-\tau)].$$

Eq. (3), using the constant proportional growth restriction and remembering that $l_{\tau-1} = G_l^{\tau-1} l_0$, then becomes

$$\lambda(t) = \lambda(0)\alpha_0^t G_\alpha^{(1/2)t(t-1)} + \sum_{\tau=1}^{t} \alpha_0^{t-\tau} G_\alpha^{(1/2)[(t^2-t)-(\tau^2-\tau)]} G_l^{\tau-1} l_0$$

$$= \lambda(0)\alpha_0^t G_\alpha^{(1/2)(t^2-t)} + l_0 \alpha_0^t G_\alpha^{(1/2)(t^2-t)} \sum_{\tau=1}^{t} \alpha_0^{-\tau} G_\alpha^{-(1/2)(\tau^2-\tau)} G_l^{\tau-1}. \qquad (7)$$

This solution is not closed-form, in that the sum in the second term does not have an algebraic counterpart. It can be verified, however, by forming $\lambda(t+1)$, substituting the first term of (7) into the first term of $\lambda(t+1)$ and simplifying – a procedure that reproduces the original difference equation (4).

We may note here that with $G_\alpha = 1$ (the neutral case) the solution (7) reduces to

$$\lambda(t) = \alpha_0^t \lambda(0) + \frac{l_0 \alpha_0^t}{G_l} \sum_{\tau=1}^{t} (G_l/\alpha_0)^\tau .$$

The sum in this expression now has a simple geometric form, and is easily calculated to be

$$\frac{G_l^t - \alpha_0^t}{\alpha_0^t} \frac{G_l}{G_l - \alpha_0},$$

Substituting this into the expression for $\lambda(t)$ above, and rearranging, we get the solution for this case, (6).

IV

The question now is: in the general case with both α and l changing, what have we got?

I will explore the properties of (7) by, first, reporting the results of numerical simulations; and, second, using (7), both observationally and numerically, to examine some properties of the path of λ for non-integer values of t.

1. The capital-"shallowing" case: $G_\alpha < 1$. A baseline simulation was run with $\alpha_0 = 0.75$, $l_0 = 0.83\dot{3}$, $G_\alpha = 0.999$, and $G_l = 0.989$.[5] Both (4) and (7) were calculated, for 100 periods; the two calculations are identical to four decimal places, suggesting that (7) is indeed an accurate solution to (4). In this and all simulations with $G_\alpha < 1$, λ falls monotonically, as long as $\lambda(0)$ is set equal to or

5. Simulations were run on the City University of New York IBM mainframe system, using the IBM language PL1. I will be happy to send the source text and examples of data obtained to anyone requesting them.

greater than its t = 0 benchmark: $\frac{l_0}{1-\alpha_0} = 3.3\dot{3}$. When $\lambda(0)$ is set below this value, $\lambda(t)$ first rises, and falls thereafter.

2. The capital-deepening case: $G_\alpha > 1$. Ignoring complementary-function effects by setting $\lambda(0) = 3.3\dot{3}$, I tried values for G_α such as 1.002, 1.003 and 1.004. With $G_\alpha = 1.002$, an interesting thing happens: λ first *falls*, and only begins to rise at t = 72. The static formulation suggests monotonically rising unit value, so long as the output/labor ratio is growing more rapidly than material input/labor, as in this case. There is, however, evidently a temporal "drag" that causes λ to fall initially. With $G_\alpha = 1.003$ the same effect is observed, except that the turning point occurs much earlier, at t = 20. When $G_\alpha = 1.004$, the temporal "drag" effect has disap-peared: λ rises continuously.[6]

3. The general solution (7) –especially the superscripts in both terms involving the expression $t^2 - t$– suggests that the behavior of λ might be more complex in the intervals between the integer values of t allowed by the discrete time formulation.[7] There is no way that time can take on any values other than integers in the original dynamic equation (4), which therefore cannot be used to study this issue. In the solution, (7), however, there are no lagged values of λ, and t can be given non-integer values in an approximation to continuous time. The only problem for this procedure concerns the summation term, which clearly can be calculated only at integer values of t. I have addressed this problem by using linear interpolation. For example, with S = the sum over τ, we have S(2) and S(3). The value S(2.5) will be calculated as S(2) + 0.5[S(3) – S(2)].

Following this procedure, I returned to the simulations and allowed t to

6. For a sustained argument concerning conditions in which technical changes with rising α may be the outcome of rational calculations by capitalists, see Laibman, 1997.

7. I note parenthetically that the expression $t^2 - t$ in (7) is possible only if time is a pure number, and that that is indeed the case for discrete time (difference) equations. The clue is that growth factors are pure numbers; so time must be as well. This only became clear to me while working out the solution (7). The concept of a period of time is completely abstract; there is no presumption of any actual unit of time (days, years, *e.g.*) being needed at all.
The use of non-integer values of *t* may seem to contradict the inherent limitation of difference equations to the case of discrete time. However, since numerical simulation of the continuous-time version of the model is extremely difficult, especially as that version also has no closed-form solution, examination of non-integer values of *t* in the discrete-time version

rise in increments of 0.1, using the same baseline parameters as in the capital-shallowing case reported above. For the integer values of t, the figures for $\lambda(t)$ are identical to those found earlier. For the fractional values, however, λ first *rises* to a maximum at $t = 0.5$ and then falls – as we would expect from $\frac{d}{dt}(t^2-t)$ (recalling here that $G_\alpha < 1$). In the intervals between 1 and 2, 2 and 3, and so on, λ again first rises, reaching maxima at $t = 1.3$, $t = 2.2$ - 2.3 (the same value to four decimal places), $t = 3.2$ and $t = 4.2$. There are thus some interesting oscillatory movements for the non-integer ranges with $t < 5$; after that, λ resumes its predicted downward movement, given $G_\alpha < 1$.

It should be noted that even in the range (0,1), the maximum of λ at $t = 0.5$, which would be suggested by $\frac{d}{dt}(t^2-t)=0$, is only approximate, due to the presence of the other factors in both terms of (7). This is obscured by the broadness of the time increment (0.1) and by the use of only four decimal places in the reported values of λ. If time is incremented by 0.01 and six decimal places are used, for example, the λ maximum in the (0,1) interval appears at $t = 0.48$.

V

Returning to the larger perspective of the introduction, we can take stock. We have explored the properties of a difficult difference equation, pointing out –originally, I believe– that those difficulties present themselves as a result of the simplest and most obvious statement of the macroeconomic value calculation problem in a dynamic context.

The central result is that temporality in value formation does not alter the most general expectations of the trend behavior of the unit value coefficient over time. λ will, for example, ultimately fall in the capital-shallowing case, and rise in the capital-deepening case. We do, however, find permanent temporal effects. First, the *level* of λ is different when time-dating is applied. Second, its *trend* may only emerge after a period of adjustment, other than the one based on initial divergence of $\lambda(0)$ from its benchmark level. Finally, λ may be subject to *oscillations* as time passes between two integer values, especially for small *t*.

Dynamic aggregate value formation is the foundation for the theory of long-period accumulation, technical change and crisis. For this reason, it is essential to get the story right, with all temporal effects taken into consideration, if only to offset fanciful claims emerging from certain quarters concerning the paradigmatic impact of "temporal" perspectives on (what is taken to be) Marx's theory of the trajectory of capitalist society.

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ABSTRACT

The unit labor value of output, at the scale of the macroeconomy, is easily calculated in a static context. In dynamics, however, with careful time-dating of inputs and outputs and continual non-neutral technical change, the difference equation for the time path of unit value does not have a closed-form solution. This solution, approached by both analytical and simulation methods, duplicates major properties of the static formulation, but it also reveals permanent temporal effects on the value path, for neutral, material-input deepening and material-input shallowing technical change, as well as some unpredicted oscillatory behaviors for non-integer values of time.